# Graph Problems Arising from Wavelength–Routing in All–Optical Networks


Bruno Beauquier*
Université de Nice–Sophia Antipolis
930 Route de Colles, BP 145
06903 Sophia Antipolis Cedex, France
bbeauqui@sophia.inria.fr

Jean-Claude Bermond*
Université de Nice–Sophia Antipolis
930 Route de Colles, BP 145
06903 Sophia Antipolis Cedex, France
bermond@diamant.unice.fr

Luisa Gargano†
Università di Salerno
Dpt di Informatica ed Applicazioni
84081 Baronissi (SA), Italy
lg@dia.unisa.it

Pavol Hell
Simon Fraser University
School of Computing Science
Burnaby, B.C., V5A 1S6, Canada
pavol@cs.sfu.ca

Stéphane Perennes*
Delft Technical University
Mekelweg 4, Kammer 07.25
26-28 CD Delft, Netherlands
perennes@math.tudelft.nl

Ugo Vaccaro†
Università di Salerno
Dpt di Informatica ed Applicazioni
84081 Baronissi (SA), Italy
uv@dia.unisa.it



## Abstract

*We survey the theoretical results obtained for wavelength routing in all–optical networks, present some new results and propose several open problems. In all–optical networks the vast bandwidth available is utilized through* wavelength division multiplexing*: a single physical optical link can carry several logical signals, provided that they are transmitted on different wavelengths. The information, once transmitted as light, reaches its destination without being converted to electronic form in between, thus reaching high data transmission rates. We consider both networks with arbitrary topologies and particular networks of practical interest.*


## 1. Introduction

**Motivation.** Optical networks offer the possibility of interconnecting hundreds to thousands of users, covering local to wide area, and providing capacities exceeding substantially those of conventional technologies. Traditional networks use the electrical form to switch signals which can be modulated electronically at a maximum bit rate of the order of 10 Gbps, while the optical fiber bandwidth is about 10 THz [39], thus several orders of magnitude higher.

Optics is thus emerging as a key technology in state of the art communication networks and is expected to dominate many applications, such as video conferencing, scientific visualization and real-time medical imaging, high–speed super-computing and distributed computing [20, 38, 45]. We refer to the books of Green [20] and McAulay [29] for a comprehensive overview of the physical theory and applications of this emerging technology.

*All–optical* (or *single–hop*, see [32]) communications networks exploit photonic technologies for the implementation of both switching and transmission functions [19]. These systems provide all source-destination pairs with end-to-end transparent channels that are identified through a wavelength and a physical path. Maintaining the signal in optical form allows for high data transmission rates in these networks since there is no conversion to and from the electronic form. Such an approach allows thus the elimination of the "electronic bottleneck" of communications networks with electronic switching.


*SLOOP, joint project I3S-CNRS/UNSA/INRIA. Work partially supported by the French GDR/PRC Project PRS and by Galileo Project.

†Work partially supported by the MURST in the framework of the 40% Project: Algoritmi, Modelli di Calcolo e Strutture Informative, and by Galileo Project.

It is widely accepted that the *wavelength–division multiplexing* (WDM) [10] approach provides means to realize high–capacity networks, by partitioning the optical bandwidth into a large numbers of channels whose rates match those of the electronic transmission [8]. It allows multiple data streams to be transferred concurrently along the same optical fiber.

**The Optical Model.** In general, a WDM optical network consists of routing nodes interconnected by point–to–point fiber-optic links, which can support a certain number of wavelengths. Due to the electromagnetic interference, the same wavelength on two input ports *must* be routed to different output ports. In this paper we consider *switched* networks with *generalized* switches, which can be based on accousto–optic filters [9], as is done in [1, 4, 37]. In this kind of networks, signals for different requests may travel on a same communication link into a node $v$ (on different wavelengths) and then exit $v$ along different links, keeping their original wavelength. Thus the photonic switch can differentiate between several wavelengths coming along a communication line and direct each of them to a different output of the switch. The only constraint on the solution is that no two paths in the network sharing the same optical link have the same wavelength assignment. In switched networks it is possible to "reuse wavelengths" [38], thus obtaining a drastic reduction on the number of required wavelengths with respect to switchless networks [1].

A switched optical network consists of interconnected nodes which can be terminals, switches, or both. Terminals send and receive signals, and switches direct their input signals to one or more of the output links. Each link is bidirectional and actually consists of a pair of unidirectional links [38].

Some authors considered topologies with single undirected fiber links carrying undirected paths [34, 1, 37, 4]. However, it has since become apparent that optical amplifiers placed on the fiber will be directed devices. Hence we model the optical network as a symmetric directed graph $G = (V(G), A(G))$, where each arc represents a point–to–point unidirectional fiber-optic link. A *request* consists of an ordered pair of nodes, and an *instance* of a set of requests. A *solution* consists of settings for the switches in the network, and an assignment of *wavelengths* to the requests, so that there is a directed path (dipath) between the nodes of each request, and that no arc will carry two different signals on the same wavelength.

The cost and feasibility of switching and amplification devices depend on the number of wavelengths they handle. One should be aware of the severe limitations that current optical technologies impose on the amount of available wavelengths per fiber. While experimental systems report large number of up to 100 wavelengths per fiber [33], current state of the art manufacturing processes restrict the number of wavelengths per fiber of commercial WDM multiplexers to as low as 4 (Pirelli), 8 (Lucent Technologies), and up to 20 (IBM). Thus our aim is to minimize the number $w$ of wavelengths used in a solution. If the number of wavelengths required to realize a set of requests is greater than the number of available wavelengths, then more than one all optical communication round are to be utilized.

We remark that optical switches do not modulate the wavelengths of the signals passing through them. If an intermediate node could change the wavelength on which a signal is transmitted, routing an instance using the minimum number of wavelengths would be equivalent to the integer multicommodity flow problem. Unfortunately, current or foreseeable technologies cannot implement such a photonic switch.

While in WDM technology a fiber link requires different wavelengths for every transmission, SDM (space division multiplexing) principle allows parallel links for a single wavelength, at an additional cost. Both techniques are combined in practice to find an efficient tradeoff between the two approaches. However, we focus our study on the directed WDM model. In fact, obtaining good bounds on the number of wavelengths provides additional evidence in favour of the WDM approach [30].

The actual process of setting up switches and routes, and of assigning wavelengths, is done using an electronic backbone control network. One may wonder at the use of a relatively slow electronic network to set up these high speed connections. In fact, the major applications for such networks require connections that last for relatively long periods once set up. Thus the initial overhead is acceptable as long as sustained throughput at high data rates is subsequently available.

**Content of the paper.** In this paper we survey the main theoretical results in the area of wavelength–routing in all–optical networks. To keep our overview as complete and current as possible, we have included new results not yet published, some of them obtained by different subsets of authors of the present paper. We have also included several open problems which we hope will stimulate further research in the area.

## 2. Definitions

There are several natural ways in which an all–optical network can be modeled. In this paper, we choose for the most part to model it as a *symmetric digraph*, that is, a directed graph with vertex set $V(G)$ and arc set $A(G)$, such that if $(u, v) \in A(G)$ then $(v, u) \in A(G)$. On the other hand, the definitions given below apply to any (symmetric or not) digraphs, and in some cases we will make comments about general digraphs. We always denote by $N$ the number of vertices in $G$, that is, $N = |V(G)|$.

We also use the following notation:

- $P(x, y)$ denotes a *dipath* in $G$ from the node $x$ to $y$, that is a directed path which consists of a set of consecutive arcs beginning in $x$ and ending in $y$.
- $\delta(x, y)$ denotes the *distance* from $x$ to $y$ in $G$, that is the length of a shortest dipath $P(x, y)$.

An algorithm will be said *efficient* if it is deterministic and runs in polynomial time. We have chosen not to consider probabilistic algorithms in this paper.

**Wavelength-routing problem**

- A *request* is an ordered pair of nodes $(x, y)$ in $G$ (corresponding to a message to be sent by $x$ to $y$).
- An *instance* $I$ is a collection of requests. Note that a given request $(x, y)$ can appear more than once in an instance.
- A *routing* $R$ for an instance $I$ in $G$ is a set of dipaths $R = \{P(x, y) \mid (x, y) \in I\}$.
- The *conflict graph* associated to a routing $R$ is the undirected graph $(R, E)$ with vertex set $R$ and such that two dipaths of R are adjacent if and only if they share an arc of G.

Let $G$ be a digraph and $I$ an instance. The *problem* $(G, I)$ asks for a routing $R$ for the instance $I$ and assigning each request $(x, y) \in I$ a wavelength, so that no two dipaths of $R$ sharing an arc have the same wavelength. If we think of wavelengths as colors, the problem $(G, I)$ seeks a routing $R$ and a vertex coloring of the conflict graph $(R, E)$, such that two adjacent vertices are colored differently. We denote by $\vec{\mathtt{w}}(G, I, R)$ the chromatic number of $(R, E)$, and by $\vec{\mathtt{w}}(G, I)$ (or briefly just $\vec{\mathtt{w}}$ if there is no ambiguity) the smallest $\vec{\mathtt{w}}(G, I, R)$ over all routings R. Thus $\vec{\mathtt{w}}(G, I, R)$ is the minimum number of wavelengths for a routing $R$ and $\vec{\mathtt{w}}(G, I)$ the minimum number of wavelengths over all routings for $(G, I)$.

**Remark 1** As explained in the introduction, early models of all–optical networks used undirected graphs, and many results are formulated in that context. In this 'undirected' model, to conflict means to share an edge. All the definitions for the directed case have natural analogues in the undirected case. Note that we use the same notation $G$ for two different objects: an undirected graph or its induced symmetric digraph obtained by replacing each edge by two opposite arcs. However, in what follows we will distinguish the directed and undirected concepts by using an arrow over the directed parameters.

**Remark 2** Any routing by undirected paths induces a routing by directed paths, and a coloring of the undirected paths is also a coloring of the directed paths, as two edge-disjoint paths will become arc-disjoint dipaths. Hence $\vec{\mathtt{w}}(G, I) \leq \mathtt{w}(G, I)$ for any problem $(G, I)$, and every upper bound on $\mathtt{w}$ is an upper bound on $\vec{\mathtt{w}}$. At first glance one could think that $\mathtt{w} \leq 2\vec{\mathtt{w}}$. That is not true as shown by the case of the instance $I = \{(1, 2), (2, 3), (3, 1)\}$ in the 3-star network $G$ where $V(G) = \{0, 1, 2, 3\}$ and $E(G) = \{\{0, 1\}, \{0, 2\}, \{0, 3\}\}$. Indeed $\vec{\mathtt{w}}(G, I) = 1$ and $\mathtt{w}(G, I) = 3$. Furthermore the ratio $\mathtt{w}/\vec{\mathtt{w}}$ may be arbitrarily great: in the class of mesh-like networks given in [1], we can have $\vec{\mathtt{w}} = 1$ and $\mathtt{w} = n$, for each positive integer $n$.

**Special instances**

- The *All–to–All* instance is $I_{\mathtt{A}} = \{(x, y) \mid x \in V(G), y \in V(G), x \neq y\}$.
- A *One–to–All* instance is a set $I_{\mathtt{O}} = \{(x_0, y) \mid y \in V(G), y \neq x_0\}$, where $x_0 \in V(G)$. A *One–to–Many* instance is a subset of some instance $I_{\mathtt{O}}$.
- A *k-relation* is an instance $I_k$ in which each node is a source and a destination of no more than $k$ requests. A 1-relation is also known as a *permutation* instance. Note also that the instance $I_A$ is an $(N - 1)$-relation.

## 3. A related parameter

- Given a network $G$ and a routing $R$ for an instance $I$, the *load* of an arc $\alpha \in A(G)$ in the routing $R$, denoted by $\vec{\pi}(G, I, R, \alpha)$, is the number of dipaths of $R$ containing $\alpha$. The *load* (also called *congestion*) of $G$ in the routing $R$, denoted by $\vec{\pi}(G, I, R)$, is the maximum load of any arc of $G$ in the routing R, that is, $\vec{\pi}(G, I, R) = \max_{\alpha \in A(G)} \vec{\pi}(G, I, R, \alpha)$.
- The *load* of $G$ for an instance $I$, denoted by $\vec{\pi}(G, I)$, or $\vec{\pi}$ if there is no ambiguity, is the minimum load of $G$ in any routing $R$ for $I$, that is, $\vec{\pi}(G, I) = \min_R \vec{\pi}(G, I, R)$. For the All–to–All instance $I_{\mathtt{A}}$, $\vec{\pi}(G, I_{\mathtt{A}})$ (respectively $\pi(G, I_{\mathtt{A}})$) is called the *arc forwarding index* (resp. *edge forwarding index*, see [24, 41]) of $G$.

The relevance of this parameter to our problem is shown by the following lemma:

**Lemma 3** *$\vec{\mathtt{w}}(G, I) \geq \vec{\pi}(G, I)$ for any instance $I$ in any network $G$.*

In other words, to solve a given problem $(G, I)$ one has to use a number of wavelengths at least equal to the maximum number of dipaths having to share an arc. The inequality can be strict, as shown by Figure 1, analogous to an example from [30].

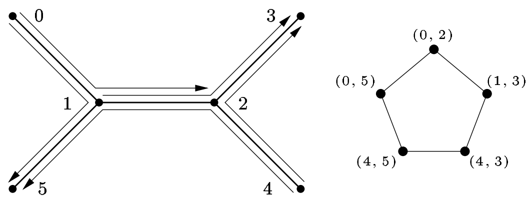


**Figure 1. A routing for five requests in a tree and its associated conflict graph.**

Indeed for this instance $I$ in the tree $G$, the load is $\vec{\pi}(G, I) = 2$ but $\vec{\mathtt{w}}(G, I) = 3$, since the conflict graph is a pentagon which has chromatic number 3.

In general, minimizing the number of wavelengths is not the same problem as that of realizing a routing that minimizes the number of dipaths sharing an arc. Indeed, our problem is made much harder due to the further requirement of wavelengths assignment on the dipaths. In order to get equality in Lemma 3, one should find a routing $R$ such that $\vec{\pi}(G, I, R) = \vec{\pi}(G, I)$, for which the associated conflict graph is $\vec{\pi}(G, I)$–vertex colorable.

**Question 4** Does there always exist a routing $R$ such that $\vec{\pi}(G, I, R) = \vec{\pi}(G, I)$ and at the same time $\vec{\mathtt{w}}(G, I, R) = \vec{\mathtt{w}}(G, I)$?

**Theorem 5** *Determining $\vec{\pi}(G, I)$ in the general case is NP–complete.*

**Sketch of proof.** We first observe that determining $\vec{\pi}(G, I)$ is equivalent to solving the integral multicommodity directed flow problem with constant capacities. It is shown in [14] that this problem is NP–complete even for two commodities and all capacities equal to one. □

For some special problems, $\vec{\pi}(G, I)$ can be efficiently determined. That is obviously the case for trees, where we always have a unique routing. That is also the case of the One–to–Many instances where the problem can be reduced to a flow problem (in the graph obtained from $G$ by considering the sender node as the source, giving a capacity $\vec{\pi}$ to each arc of $G$, and joining all the vertices of $G$ to a sink $t$ with arcs of capacity 1).

**Remark 6** We can define analogously the load $\pi(G, I)$ for an undirected graph and we can prove that $\vec{\pi}(G, I) \leq \pi(G, I) \leq 2\vec{\pi}(G, I)$. For One–to–Many instances $I$, we can also show that $\vec{\pi}(G, I) = \pi(G, I)$.

**Question 7** Does the equality $\vec{\pi}(G, I_{\mathtt{A}}) = \lceil \pi(G, I_{\mathtt{A}})/2 \rceil$ always hold ?

The following question was also asked in [24]:

**Question 8** What is the complexity of determining $\vec{\pi}(G, I_{\mathtt{A}})$ ?

Let the *arc expansion* $\beta(G)$ of a directed graph $G$ having $N$ nodes be the minimum, over all subsets of nodes $S \subset V(G)$ of size $|S| \leq N/2$, of the ratio of the number of arcs with origin in $S$ and destination outside of $S$, to the size of $S$. It follows from [41] that $\vec{\pi}(G, I_{\mathtt{A}}) \geq \frac{N}{2\beta(G)}$.

## 4. Arbitrary networks

### 4.1. Arbitrary instances

For a general network $G$ and an arbitrary instance $I$, the problem of determining $\vec{\mathtt{w}}(G, I)$ has been proved to be NP–hard in [12]. In particular, it has been proved that determining $\vec{\mathtt{w}}(G, I)$ is NP–hard for trees and cycles. In [13] these results have been extended to binary trees and meshes. NP–completeness results in the undirected model were known much earlier (actually, well before the advent of the WDM technology). In particular, in [18] it is proved that the problem of determining $\mathtt{w}(G, I)$ is NP–complete for trees. This result has been extended in [12] to cycles, while in [13] it has been proved that the problem is efficiently solvable for bounded degree trees.

In view of this last result and of the NP–hardness of determining $\vec{\mathtt{w}}(G, I)$ for binary trees, it might seem that the problem of computing $\vec{\mathtt{w}}(G, I)$ is harder than that of computing $\mathtt{w}(G, I)$. This is not true in general. For instance, the determination of $\mathtt{w}(G, I)$ remains NP–complete when $G$ is a star network, whereas $\vec{\mathtt{w}}(G, I)$ can be efficiently computed. Indeed, in the undirected model this problem corresponds to edge-coloring a multigraph [25], each node of which corresponds to a branch in the star network. In the directed case, the same problem becomes equivalent to edge-coloring a bipartite multigraph, efficiently solvable by König's theorem.

In [1] is given an upper bound in the undirected model, which also holds in the directed case:

**Theorem 9** (Aggarwal et al. [1]). *For any problem $(G, I)$, where $G$ has $m$ arcs, $\vec{\mathtt{w}}(G, I) \leq 2\vec{\pi}(G, I)\sqrt{m}$.*

Let $R$ be a routing for an instance $I$ in a network $G$. Let $L$ be the maximum length of its dipaths and $\Delta$ the maximum degree of its conflict graph. It is clear that $\Delta \leq L\vec{\pi}(G, I, R)$. By a greedy coloring we know that $(\Delta + 1)$ wavelengths are sufficient to solve the problem $(G, I)$. Thus $\vec{\mathtt{w}} = O(L\vec{\pi})$ and similarly $\mathtt{w} = O(L\pi)$. A set of critical undirected problems reaching asymptotically this upper bound (and that of Theorem 9) has been given in mesh-like networks (see [1]). By adapting their examples (orienting alternately the vertical links up and down), we obtain the same result in general (not symmetric) digraphs:

**Theorem 10** *For every* $\pi$ *and* $L$*, there exists a directed graph* $G$ *and an instance* $I$ *such that* $\vec{\pi}(G,I) = \pi$, $L = \max_{(x,y)\in I} \delta(x,y)$ *and* $\vec{w}(G,I) = \Omega(\pi L)$.

**Question 11** Does Theorem 10 hold for symmetric digraphs?

### 4.2. Permutations and $k$-relations

By deriving a lower bound on the number of links used in the worst case and an upper bound on the total number of links in the network, Pankaj [34, 35] obtained in his thesis results in the undirected model that are easy to translate in the directed model:

**Theorem 12** *For every symmetric digraph* $G$ *of maximum degree* $\Delta$*, there exists a worst case permutation instance* $I_1$ *such that*

$$\vec{w}(G,I_1) \geq \frac{\lfloor \log_\Delta \frac{N}{2} \rfloor}{2\Delta} \ .$$

**Theorem 13** *For every vertex transitive symmetric digraph* $G$ *of diameter* $D$ *and degree* $\Delta$*, there exists a worst case permutation instance* $I_1$ *such that*

$$\vec{w}(G,I_1) \geq \left\lceil \frac{D}{\Delta} \right\rceil \ .$$

In addition, Pankaj obtained the lower bound $(\min\{k,N/2\}.\lfloor \log_\Delta \frac{N}{2} \rfloor / 2\Delta)$ for a worst case $k$-relation $I_k$, showing in terms of growth rate that the necessary number of wavelengths is $\Omega(\frac{k \log N}{\Delta \log \Delta})$.

In the undirected model, Raghavan and Upfal [37] have shown an existential lower bound which relates the number of wavelengths and the edge expansion, by starting from the same example that we can find in [1]. In the same way that we have obtained Theorem 10, we obtain an existential lower bound for digraphs:

**Theorem 14** *For every* $\beta \leq 1$ *and* $1 \leq k \leq N$*, there exists a problem* $(G,I_k)$ *for a* $k$*-relation* $I_k$ *in a digraph* $G$ *with arc expansion* $\beta$*, such that*

$$\vec{w}(G,I_k) = \Omega(k/\beta^2) \ .$$

**Question 15** Does Theorem 14 hold for symmetric digraphs?

Finding a routing $R$ for an instance $I$ in a network $G$ and minimizing the load $\vec{\pi}(G,I,R)$ of $G$ in $R$ corresponds to an integer multicommodity flow problem. When $I$ is a permutation instance, Leighton and Rao (see Theorem 2 in [28]) have given an efficient algorithm to obtain an integer flow with maximum load and dilatation (length of a longest dipath) both $O(\log N/\beta(G))$. Since each vertex in the conflict graph of the resulting routing has degree upper bounded by (load × dilation), the greedy coloring algorithm allows to use $O(\log^2 N/\beta^2)$ colors, as it has been noticed in [4]. By Hall's theorem, this implies a result for $k$-relations in the directed model:

**Theorem 16** *There is an efficient algorithm to solve the problem* $(G,I)$ *for every* $k$*-relation* $I_k$ *in any bounded degree network* $G$ *with arc expansion* $\beta$*, using at most* $O(k \log^2 N/\beta^2)$ *wavelengths.*

Note that this result almost matches the $\Omega(k/\beta^2)$ existential lower bound of Theorem 14. We can also put $\vec{w}(G,I)$ in relation with the *arc connectivity* $\lambda$ of $G$. A digraph $G$ has arc connectivity $\lambda$ if the minimum number of arcs to remove in order to disconnect $G$ is $\lambda$. Using a theorem of Shiloach [40], we can prove the following result:

**Theorem 17** *There is an efficient algorithm to solve the problem* $(G,I)$ *for every instance* $I$ *in any symmetric digraph* $G$ *with arc connectivity* $\lambda$*, using at most* $\lceil |I|/\lambda \rceil$ *wavelengths. Moreover, this bound is best possible for worst case instances.*

### 4.3. Other specific instances

The following theorem gives the exact value of $\vec{w}(G,I_0)$ for a worst case instance $I_0$ in various classes of important networks, namely the *maximally arc connected* digraphs, including the wide class of vertex transitive digraphs. A digraph $G$ is maximally arc connected if its minimum degree is equal to its arc connectivity.

**Theorem 18** (Bermond et al. [7]). *For a worst case One–to–All instance* $I_0$ *in a maximally arc connected digraph* $G$ *of minimum degree* $d(G)$*,*

$$\vec{w}(G,I_0) = \vec{\pi}(G,I_0) = \left\lceil \frac{N-1}{d(G)} \right\rceil \ .$$

In addition, an efficient network flow based algorithm is given to solve the problem $(G,I)$ with $\vec{w}(G,I)$ wavelengths, for any One–to–Many instance $I$ in any network $G$. We have recently generalized the last theorem:

**Theorem 19** $\vec{w}(G,I) = \vec{\pi}(G,I)$*, for any One–to–Many instance* $I$ *in any digraph* $G$*.*

We have the following question. Two of the present authors believe the answer is positive, other two believe it is negative and the last two carefully abstain.

**Question 20** Does the equality $\vec{w}(G,I_A) = \vec{\pi}(G,I_A)$ hold for the All–to–All instance $I_A$ in any symmetric digraph $G$?

## 5. Specific networks

### 5.1. Trees

The case of trees is particularly interesting as many practical networks, e.g., in the telecommunications industry, have a tree-like structure (see [30]). Let us consider first the case when the network $G$ is a symmetric subdivided star, that is, when $G$ is a symmetric tree with at most one node with outdegree greater than 2. In this case, for every instance $I$, $\vec{w}(G,I) = \vec{\pi}(G,I)$. Actually, when $G$ is a path this is equivalent to the fact that the chromatic number of an interval graph is equal to the maximum size of its cliques. When $G$ is a star this is equivalent to the fact that in a bipartite graph the edge chromatic index is equal to its maximum degree. When $G$ is a subdivided star, we can combine these two approaches. It is not difficult to observe that the converse also holds, i.e., when $T$ is a symmetric tree other than a subdivided star, then there exists an instance $I$ such that $\vec{w}(G,I) \neq \vec{\pi}(G,I)$:

**Theorem 21** *Let $G$ be a symmetric tree. Then for all instances $I$ $\vec{w}(G,I) = \vec{\pi}(G,I)$ if and only if $G$ is a subdivided star.*

In fact, a tree other than a subdivided star is a subdivision of the graph shown in Figure 1, thus we can always choose requests such that the conflict graph is a pentagon; thus $\vec{w} = 3$, $\vec{\pi} = 2$. It was generally believed [30] until very recently that in a symmetric tree the ratio $\vec{w}/\vec{\pi}$ was never greater than $3/2$. However, Jansen [26] constructed a problem $(G,I)$ (where $G$ is a symmetric tree) with $\vec{w} = 5$, $\vec{\pi} = 3$. Moreover, the following result has been obtained by Kaklamanis and Persiano and subsequently found by Erlebach and Jansen:

**Theorem 22** ([27] and [11]). *Let $G$ be a symmetric tree. Then for all instances $I$ we have $\vec{w}(G,I) \leq 5\vec{\pi}(G,I)/3$, and there is an efficient algorithm to find a $5\vec{\pi}(G,I)/3$-coloring.*

We now show that in a tree network there exist problems with arbitrarily great load $\vec{\pi}$, such that the ratio $\vec{w}/\vec{\pi}$ is greater than $5/4$. We start from the problem $(G,I)$ shown in Figure 1. For every natural number $n$, let $I^n$ denote the instance made of $n$ copies of $I$ (each request of $I$ is repeated $n$ times). Coloring the conflict graph associated to the problem $(G,I^n)$ becomes then a multicoloring problem for the pentagon, and from what is known about this problem (see [23]) we obtain:

**Theorem 23** *For every $\pi$ there exists a problem $(G,I)$ in a tree $G$, such that $\vec{\pi}(G,I) \geq \pi$ and $\vec{w}(G,I) \geq 5\vec{\pi}(G,I)/4$.*

**Question 24** Can the constant of Theorem 23 be raised?

Note that Theorem 22 implies an approximation algorithm to solve the problem $(G,I)$ in a symmetric tree G with at most $5\vec{w}(G,I)/3$ wavelengths. For undirected graphs, there is a better result:

**Theorem 25** (Erlebach and Jansen [12]). *There is an efficient algorithm to solve the problem $(G,I)$ for any instance $I$ in any tree network $G$, using at most $\lfloor 1.1\mathtt{w}(G,I) + 0.8 \rfloor$ wavelengths.*

In the case of subdivided stars above the problem of determining $\vec{w}$ can be efficiently solved, since that is the case for the chromatic number of interval graphs and the chromatic index of bipartite graphs.

In the case of undirected trees $G$, Tarjan [43] proved that $\mathtt{w}(G,I) \leq 3\pi(G,I)/2$, and this bound is achieved in the example of Remark 2. Edge-coloring of multigraphs is an NP–complete problem [25], and since to each multigraph corresponds the conflict graph of some instance $I$ in some star $G$, the computation of $\mathtt{w}$ in stars (and hence trees) is NP–complete.

**Theorem 26** (Gargano, Hell and Perennes [17]). *For the All-to-All instance $I_{\mathtt{A}}$ in any symmetric tree $G$ we have $\vec{w}(G,I) = \vec{\pi}(G,I)$, and there is an efficient algorithm to find a $\vec{\pi}(G,I)$-coloring.*

In the undirected model, there are examples where the ratio $\mathtt{w}(G,I_{\mathtt{A}})/\pi(G,I_{\mathtt{A}})$ can tend asymptotically to $3/2$. For instance, it is the case for the family of trees having three branches of equal size.

**Question 27** Can the problem $(G,I_{\mathtt{A}})$ for every undirected tree $G$ be efficiently solved, using exactly $\mathtt{w}(G,I_{\mathtt{A}})$ wavelengths?

Golumbic and Jamison [18] proved that the conflict graph of a set of undirected paths in a tree satisfies the strong perfect graph conjecture, and that the problem of finding a clique of maximum cardinality can be efficiently solved.

To our knowledge, the particular case of permutation instances in tree networks has not been studied in the literature. By adapting the example shown in Figure 1, it follows that we can have $\vec{w}(G,I_1) = 3\vec{\pi}(G,I_1)/2$ for a permutation instance $I_1$ in a tree network $G$.

Finally, it is worth pointing out that when $G$ is an oriented tree (each edge oriented in exactly one way), Monma and Wei [31] have proved that for any instance $I$, we have $\vec{w}(G,I) = \vec{\pi}(G,I)$ and $\vec{w}(G,I)$ can be efficiently computed.

### 5.2. Rings, tori and meshes

**Theorem 28** (Frank et al. [15]). *There is an linear time algorithm to find a routing $R$ for any instance $I$ in any undirected ring network $G$, such that $\pi(G,I,R) = \pi(G,I)$.*

**Question 29** Does Theorem 28 also hold in the directed model ?

A routing being fixed in a ring network, the wavelength assignment becomes in both models a vertex coloring of circular arc graph, which in the general case is proved to be NP–complete in [16]. As observed earlier, also the general problem in both models of determining $\vec{\mathtt{w}}$ or $\mathtt{w}$ is NP–hard for ring networks (see [12]). Nevertheless there are some approximation results.

Given a routing $R$ for an instance $I$ in a ring $G$, Tucker [44] gave an efficient algorithm to solve the wavelengths assignment problem, using at most $(2\vec{\pi}-1)$ wavelengths. Combined with Theorem 28, this result gives an efficient approximation algorithm of ratio two for the problem $(G, I)$ in undirected ring networks $G$. Using the same idea as Tucker, such approximation algorithms have been shown in the undirected model in [37] and in the directed model in [30]. In addition, Tucker showed examples with arbitrarily great $\vec{\pi}$ necessitating the use of $(2\vec{\pi}-1)$ wavelengths. For instance, it is the case for the five distinct requests $(x, x+2 \bmod 5)$ in the $5$-ring.

The following result is the first *per-instance* approximation algorithm for bounded dimension meshes and it also holds for bounded dimension tori. We give here its directed version.

**Theorem 30** (Aumann and Rabani [4]). *There is an efficient algorithm to solve the problem $(G, I)$ for any instance $I$ in any bounded dimension mesh network $G$, using at most $O(\log N \log |I| \mathtt{w}(G, I))$ wavelengths.*

Rabani [36] improved recently the previous approximation results obtained for the square meshes, although the hidden constants are huge:

**Theorem 31** (Rabani [36]).*There is an efficient algorithm to solve the problem $(G, I)$ for any instance $I$ in any square mesh network $G$, using at most* $\mathrm{poly}(\log\log N).\vec{\mathtt{w}}(G, I)$ *wavelengths, where* poly *denotes a polynomial function.*

This result also holds in the directed model and it has been given in addition an efficient algorithm to determine $\mathtt{w}$ in square meshes with a constant approximation ratio.

Regarding the All–to–All instance $I_{\mathtt{A}}$, the following two theorems extend results of [2]:

**Theorem 32** (Bermond et al. [7]).*For the All–to–All instance $I_{\mathtt{A}}$ in the ring network $G$ with $N$ nodes, $\vec{\mathtt{w}}(G, I_{\mathtt{A}}) = \vec{\pi}(G, I_{\mathtt{A}}) = \lceil \lfloor N^2/4 \rfloor /2 \rceil$.*

**Theorem 33** (Beauquier [6]). *For the All–to–All instance $I_{\mathtt{A}}$ in the $d$-dimensional hypersquare torus $G$ with even side, $\vec{\mathtt{w}}(G, I_{\mathtt{A}}) = \vec{\pi}(G, I_{\mathtt{A}}) = N^{\frac{d+1}{d}}/8$.*

### 5.3. Hypercubes

Following the idea of [4], Gu and Tamaki [21] proved that $8$ wavelengths are sufficient to realize a permutation in undirected hypercubes. In the directed model, they obtained the following theorem, not far from Szymanski's conjecture [42] stating that one wavelength is sufficient:

**Theorem 34** (Gu and Tamaki [22]). *For any permutation instance $I_1$ in any symmetric directed hypercube $G$, the problem $(G, I_1)$ can be efficiently solved with $2$ wavelengths.*

For the All–to–All instance $I_{\mathtt{A}}$, the following result uses the idea developped for hypercubes in [34] and independently in [7]. It also follows from the study of compound graphs in [3].

**Theorem 35** (Beauquier [6]). *For the All–to–All instance $I_{\mathtt{A}}$ in any cartesian sum $G$ of complete graphs, the problem $(G, I_{\mathtt{A}})$ can be efficiently solved with $\vec{\mathtt{w}}(G, I_{\mathtt{A}}) = \vec{\pi}(G, I_{\mathtt{A}})$ wavelengths.*

## 6. Final remarks

We gave a survey of the main theoretical results arising from wavelength–routing in all–optical networks and posed several questions for future research. Hopefully we have shown the reader how graph theoretic tools can help in the design of all–optical networks.

Finally, it is worth pointing out that another very important line of research is that of on–line routing in optical networks. In this scenario, requests can dynamically change and are given at different times. We refer to [5] and references therein for an account of this area.

## Acknowledgements

We are grateful to Pr. Paulraja for fruitful discussions and remarks.

## References

[1] A. Aggarwal, A. Bar-Noy, D. Coppersmith, R. Ramaswami, B. Schieber, and M. Sudan. Efficient routing and scheduling algorithms for optical networks. In *Proc. of the 5th Symposium on Discrete Algorithms (SODA)*, pages 412–423. ACM-SIAM, Jan. 1994.

[2] M. Ajmone Marsan, A. Bianco, E. Leonardi, and F. Neri. Topologies for wavelength–routing all-optical networks. *IEEE/ACM Trans. on Networking*, 1(5):534–546, 1993.

[3] D. Amar, A. Raspaud, and O. Togni. Échange total dans les réseaux optiques obtenus par composition de cliques. Manuscript, 1996.

[4] Y. Aumann and Y. Rabani. Improved bounds for all optical routing. In *Proc. of the 6th Symposium on Discrete Algorithms (SODA)*, pages 567–576. ACM-SIAM, Jan. 1995.

[5] B. Awerbuch, Y. Azar, A. Fiat, and S. Leonardi. On-line competitive algorithms for call admission in optical networks. *Lecture Notes in Computer Science*, 1136:431–444, 1996.

[6] B. Beauquier. All to all communication for wavelength routed all optical networks. Manuscript, 1996.

[7] J.-C. Bermond, L. Gargano, S. Perennes, A. A. Rescigno, and U. Vaccaro. Efficient collective communication in optical networks. *Lecture Notes in Computer Science*, 1099:574–585, 1996.

[8] C. A. Brackett. Dense wavelength division multiplexing networks: principles and applications. *IEEE Journal on Selected Areas in Communications*, 8(6):948–964, 1990.

[9] K.-W. Cheng. Accousto-optic tunable filters in narrowband WDM networks. *Journal on Selected Areas in Communications*, 8:1015–1025, 1990.

[10] N. K. Cheung, N. K., and G. Winzer. Special issue on dense WDM networks. *Journal on Selected Areas in Communications*, 8, 1990.

[11] T. Erlebach and K. Jansen. An optimal greedy algorithm for wavelength allocation in directed tree networks. Manuscript.

[12] T. Erlebach and K. Jansen. Scheduling of virtual connections in fast networks. In *Proc. of Parallel Systems and Algorithms (PASA)*, pages 13–32, 1996.

[13] T. Erlebach and K. Jansen. Call scheduling in trees, rings and meshes. In *Proc. of HICSS*, 1997.

[14] S. Even, A. Itai, and A. Shamir. On the complexity of timetable and multicommodity flow problems. *SIAM Journal of Computing*, 5(4):691–703, Dec. 1976.

[15] A. Frank, T. Nishizeki, N. Saito, H. Suzuki, and E. Tardos. Algorithms for routing around a rectangle. *Discrete Applied Mathematics*, 40:363–378, 1992.

[16] M. R. Garey, D. S. Johnson, G. L. Miller, and C. H. Papadimitriou. The complexity of coloring circular arcs and chords. *SIAM Journal on Algebraic and Discrete Methods*, 1(2):216–227, June 1980.

[17] L. Gargano, P. Hell, and S. Perennes. Coloring all directed paths in a symmetric tree. Manuscript, 1996.

[18] M. C. Golumbic and R. E. Jamison. The edge intersection graphs of paths in a tree. *Journal of Combinatorial Theory, Series B*, 38:8–22, 1985.

[19] P. E. Green. The future of fiber–optic computer networks. *IEEE Computer*, 24(9):78–87, Sept. 1991.

[20] P. E. Green. *Fiber–Optic Communication Networks*. Prentice–Hall, 1993.

[21] Q.-P. Gu and H. Tamaki. Edge-disjoint routing in the undirected hypercube. Technical Report COMP 96-21, Institute of Electronics, Information, and Communication Engineering of Japan, 1996.

[22] Q.-P. Gu and H. Tamaki. Routing a permutation in the hypercube by two sets of edge-disjoint paths. In *Proc. of 10th IPPS*. IEEE Computer Society Press, 1996.

[23] P. Hell and F. S. Roberts. Analogues of Shannon capacity. *Annals of Discrete Mathematics*, pages 155–168, 1982.

[24] M.-C. Heydemann, J.-C. Meyer, and D. Sotteau. On forwarding indices of networks. *Discrete Applied Mathematics*, 23:103–123, 1989.

[25] I. Holyer. The NP–completeness of edge coloring. *SIAM Journal of Computing*, 10(4):718–720, 1981.

[26] K. Jansen. Approximation results for wavelength routing in directed trees. In *Proc. of WOCS*, 1997.

[27] C. Kaklamanis and G. Persiano. Constrained bipartite edge coloring with applications to wavelength routing. Manuscript, 1996.

[28] F. T. Leighton and S. Rao. An approximate max-flow mincut theorem for uniform multicommodity flow problems with applications to approximation algorithms'. In *Proc. of 29th FOCS*, pages 422–431. IEEE, Oct. 1988.

[29] A. D. McAulay. *Optical Computer Architectures*. John Wiley, 1991.

[30] M. Mihail, C. Kaklamanis, and S. Rao. Efficient access to optical bandwidth—wavelength routing on directed fiber trees, rings, and trees of rings. In *Proc. of 36th FOCS*, pages 548–557. IEEE, Oct. 1995.

[31] C. L. Monma and V. K. Wei. Intersection graphs of paths in a tree. *Journal of Combinatorial Theory, Series B*, pages 141–181, 1986.

[32] B. Mukherjee. WDM-based local lightwave networks, Part 1: Single-hop systems. *IEEE Network Magazine*, 6(3):12–27, May 1992.

[33] K. Nosu, H. Toba, K. Inoue, and K. Oda. 100 channel optical FDM technology and its applications to optical FDM channel-based networks. *IEEE/OSA Journal of Lightwave Theory*, 11:764–776, 1993.

[34] R. K. Pankaj. *Architectures for Linear Lightwave Networks*. PhD thesis, Dept. of Electrical Engineering and Computer Science, MIT, Cambridge, MA, 1992.

[35] R. K. Pankaj and R. G. Gallager. Wavelength requirements of all–optical networks. *IEEE/ACM Trans. on Networking*, 3:269–280, 1995.

[36] Y. Rabani. Path coloring on the mesh. In *Proc. of 37th FOCS*. IEEE, Oct. 1996.

[37] P. Raghavan and E. Upfal. Efficient routing in all-optical networks. In *Proc. of the 26th STOC*, pages 134–143. ACM, May 1994.

[38] R. Ramaswami. Multiwavelength lightwave networks for computer communication. *IEEE Communications Magazine*, 31(2):78–88, Feb. 1993.

[39] M. Settembre and F. Matera. All optical implementations of high capacity TDMA networks. *Fiber and Integrated Optics*, 12:173–186, 1993.

[40] Y. Shiloach. Edge–disjoint branching in directed multigraphs. *Information Processing Letters*, 8(1):24–27, 1979.

[41] P. Solé. Expanding and forwarding. *Discrete Applied Mathematics*, 58:67–78, 1995.

[42] T. Szymanski. On the permutation capability of a circuit-switched hypercube. In *Proc. of Int. Conf. on Parallel Processing (ICPP)*, vol. 1, pages 103–110, Aug. 1989.

[43] R. E. Tarjan. Decomposition by clique separators. *Discrete Mathematics*, 55(2):221–232, 1985.

[44] A. Tucker. Coloring a family of circular arcs. *SIAM Journal of Applied Mathematics*, 29(3):493–502, 1975.

[45] R. J. Vetter and D. H. C. Du. Distributed computing with high-speed optical networks. *IEEE Computer*, 26(2):8–18, Feb. 1993.