Basic Elements for the Performance Evaluation of Networks

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0.1 Example of a discrete-time Markov chain

Consider an homogeneous discrete-time Markov chain (D-MC) on the state-space $\mathcal{E} = \{1, 2, 3\}$ with transition matrix $\mathbf{P}$ given by

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1}$$

One often represents a D-MC through its transition diagram. In this case it is given by

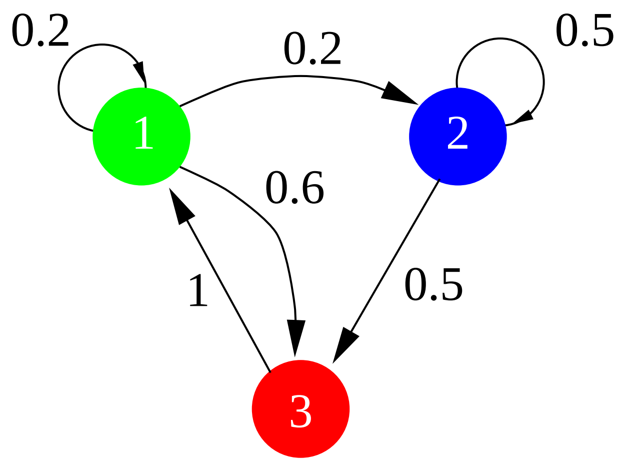

Fig. 1 – Transition diagram

The first type of question we may want to address is : *given that the chain is in state* 1 *at time* $t = 0$*, in what state will it be at time* $t = 4$ *?* There is no deterministic answer to this question since, by construction, at time $t = 4$ the chain can a priori be in any of the states 1, 2 and 3. A more relevant question is : *given that the chain is in state* 1 *at time* $t = 0$*, what is the probability that it will be in state* j *at time* $t = 4$ *?*. The answer to the latter question is obtained using formula (10) (Section 2.1. of the Lecture Notes), which will allow us to compute $\pi_4 = (\pi_4(1), \pi_4(2), \pi_4(3))$, where we recall that is $\pi_n(i)$ the probability that the chain is in state $i \in \mathcal{E}$ at time $t = n$. Here, the initial probability distribution π_0 is given by

$$\pi_0 = (1, 0, 0)$$

since we have specified that the chain is initially in state 1. We find

$$\pi_1 = (1, \quad 0, \quad 0) \begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{pmatrix} = (0.2, 0.2, 0.6)$$

$$\pi_2 = (0.2, 0.2, 0.6) \begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{pmatrix} = (0.64, 0.14, 0.22)$$

$$\pi_3 = (0.64, 0.14, 0.22) \begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{pmatrix} = (0.348, 0.198, 0.454)$$

$$\pi_4 = (0.348, 0.198, 0.454) \begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{pmatrix} = (0.5236, 0.1686, 0.3078).$$

In summary, at time $t = 4$, the chain will be

– in state 1 with probability 0.5236
– in state 2 with probability 0.1686
– in state 3 with probability 0.3078.

Let us now compute the stationary probabilities. First, note that the chain is clearly irreducible and aperiodic. Hence (cf. Proposition 3), if we find a unique strictly positive solution to the system of equations

$$\begin{cases} x_1 &= 0.2x_1 + 1.0x_3 \\ x_2 &= 0.2x_1 + 0.5x_2 \\ x_3 &= 0.6x_1 + 0.5x_3 \\ 1 &= x_1 + x_2 + x_3 \end{cases}$$

then this solution will be the stationary distribution of this Markov chain, that is, $\lim_{n\to\infty} \pi_n(i)$ will exist and will be given by x_i for $i \in \mathcal{E}$.

Solving for this system of equations gives

$$x = (5/11, 2/11, 4/11) \;\simeq\; (0.4545, 0.1818, 0.3636). \tag{2}$$

This solution being unique and strictly positive we conclude that this is the stationary probability of the chain, namely, $\lim_{n\to\infty} \pi_n(1) = 5/11$, $\lim_{n\to\infty} \pi_n(2) = 2/11$ and $\lim_{n\to\infty} \pi_n(3) = 4/11$.

In Figure 2 we have simulated the trajectory of the chain for 50 time units (starting in state 3 in red). The first bar is the state of the chain at $t = 1$, the second at $t = 2$, etc.

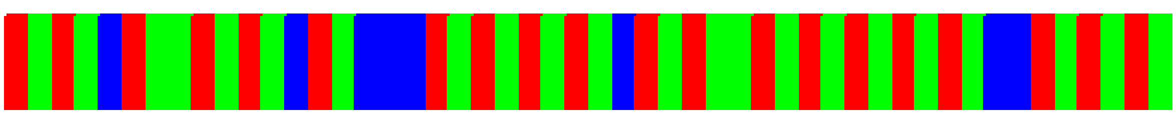

FIG. 2 – Trajectory after 50 units of time

Note that the chain does not stay more that 1 time unit in state 3 (in red) which is normal since the probability of looping back to state 3 is zero (i.e. $p_{3,3} = 0$). More precisely, upon leaving state 3 (red) the chain always goes to state 1 (green) as $p_{3,1} = 1$ by definition of $\mathbf{P}$ in (1). We also observe that the state may stay several units of time in state 1 (green) and in state 2 (blue) but that it has the tendancy to stay more often in state 2 (blue). This is also normal since $p_{2,2}$ is larger than $p_{3,3}$.

If we count the number of times that the chain has visited each state we find

$$22 \text{ times in state } 1, \quad 8 \text{ times in state } 2, \quad 20 \text{ times in state } 3$$

that is, in percentage of time it has stayed

$$44\ \% \text{ in state } 1,\ 16\ \% \text{ in state } 2,\ 40\ \% \text{ in state } 3$$

which is already close to the stationary probability found in (2).

If we observe the same trajectory for 100 units of time we obtain

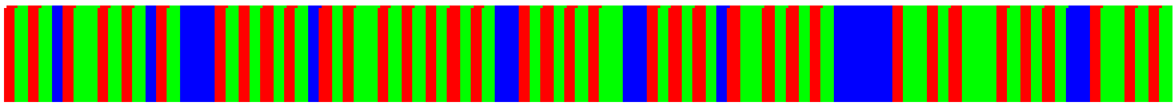

Fig. 3 – Trajectory after 100 units of time

that is the chain has visited

$$45 \text{ times state } 1, \quad 18 \text{ times state } 2, \quad 37 \text{ times state } 3$$

that is, in percentage of time it has stayed

$$45\ \% \text{ in state } 1, \quad 18\ \% \text{ in state } 2, \quad 37\ \% \text{ in state } 3$$

which is even closer from (2). There is actually a theoretical result saying that the transient probability (i.e. π_n) converges geometrically fast to the stationary probability (i.e. π) as $n \to \infty$. This means that the convergence is fast *in terms* of units of time but not necessarily in terms of real time (it may be long when the unit of time is one year and very short if the unit of time is 1 nanosecond).

It should be clear that another experiment will give totally different results in Figures 2 and 3. For instance, starting from state 1 the chain may stay in state 1 for 50 consecutive units of time with the probability $(0.2)^{50}$. This is highly unlikely (!) but not impossible.