UBINET: Performance Evaluation of Networks

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Academic Year 2010-2011

Solution of problems in list No. 1

Problem no. 1

$$
\begin{aligned}
P(\min(X_1, X_2) > x) &= P(X_1 > x\,,\, X_2 > x) \\
&= P(X_1 > x)\; P(X_2 > x) \qquad \text{independence of } X_1 \;\&\; X_2 \\
&= \begin{cases} e^{-\lambda_1 x} \cdot e^{-\lambda_2 x} & x \geq 0 \\ 0 & x < 0 \end{cases}
\end{aligned}
$$

Hence

$$
P(\min(X_1, X_2) \leq x) = \begin{cases} 1 - e^{-(\lambda_1+\lambda_2)x} & x \geq 0 \\ 0 & x < 0\,. \end{cases}
$$

$\min(X_1, X_2)$ is an exponential r.v. with parameter $\lambda_1 + \lambda_2$.

$$
P(X + 1 + X_2 \leq x) = 0 \quad \text{if} \quad x < 0
$$

$$
\begin{aligned}
P(\max(X_1, X_2) \leq x) &= P(X_1 \leq x\,,\, X_2 \leq x) \\
&= P(X_1 \leq x)\; P(X_2 \leq x) \quad \text{independence} \quad X_1 \;\&\; X_2 \\
&= \begin{cases} (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 x}) & x \geq 0 \\ 0 & x < 0\,. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
P(X_1 + X_2 \leq x) &= \int_0^x P(X_1 \leq x - y \,|\, X_2 = y)\, \lambda_2 e^{-\lambda_2 y} dy \quad \text{for} \quad x \geq 0 \\
&= \int_0^x P(X_1 \leq x - y)\, \lambda_2 e^{-\lambda_2 y} dy \quad \text{ind. of} \quad X_1 \;\&\; X_2 \\
&= \int_0^x \left(1 - e^{-\lambda_1 (x-y)}\right) \lambda_2 e^{-\lambda_2 y} dy
\end{aligned}
$$

$$= \begin{cases} 1 - e^{-\lambda_2 x} - \lambda_2 x e^{-\lambda_1 x} & \text{if} \quad \lambda_1 = \lambda_2 \\ 1 - \dfrac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 x} - \dfrac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 x} & \text{if} \quad \lambda_1 \neq \lambda_2 \end{cases}$$

Problem no. 2

- $$\frac{(30)^3}{3\,!}\; e^{-30}$$
- $$\sum_{i=1}^{20} \frac{(120)^i}{i\,!}\; e^{-120}$$
- $$\sum_{i=3}^{7} \frac{(7.5)^i}{i\,!}\; e^{-7.5}.$$

Problem no. 3

$$\begin{aligned} ES &= E \sum_{n=1}^{N} X_n \\ &= \sum_{i=1}^{\infty} E \left(\sum_{n=1}^{N} X_n \,|\, N = i \right) P\,(N = i) \\ &= \sum_{i=1}^{\infty} E \left(\sum_{n=1}^{i} X_n \right) P\,(N = i) \quad \text{independence of} \quad N \;\&\; (X_1, X_2, \ldots) \\ &= E\,(X_n) \sum_{i=1}^{\infty} i\, P\,(N = i) \;=\; M\, E[N]\;. \end{aligned}$$

Problem no. 4

$$X_n \,:\, \text{wheather condition day } n\,;\; X_n \;\in\; \{ \underset{\text{rain}}{\overset{}{\underset{\uparrow}{r}}}\;,\; \underset{\text{sun}}{\underset{\uparrow}{s}} \}$$

State $\qquad (X_{n-1}\,,\; X_n) = Z_n$

$$\begin{aligned} \cdot \quad & P(Z_{n+1} = (a,b) \,|\, Z_n = (c,d)) &=& \; 0 \quad \text{if} \quad a \neq d \\ \cdot \quad & P(Z_{n+1} = (r,r) \,|\, Z_n = (r,r)) &=& \; 0.8 \\ \cdot \quad & P(Z_{n+1} = (s,r) \,|\, Z_n = (r,s)) &=& \; 0.2 \\ \cdot \quad & P(Z_{n+1} = (r,r) \,|\, Z_n = (s,r)) &=& \; 0.5 \\ \cdot \quad & P(Z_{n+1} = (s,r) \,|\, Z_n = (s,s)) &=& \; 0.1 \end{aligned}$$

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0.9 \end{pmatrix} \begin{matrix} (r,r) \\ (r,s) \\ (s,r) \\ (s,s) \end{matrix}$$

$$(r,r) \quad (r,s) \quad (s,r) \quad (s,s)$$

Problem no. 5

$$\star \quad \pi(1) = \frac{21}{71}\,,\ \pi(2) = \frac{26}{71}\,,\ \pi(3) = \frac{24}{71}$$

$$\star \quad P^{(2)} = P^2 = \begin{pmatrix} 0.32 & 0.38 & 0.30 \\ 0.30 & 0.37 & 0.33 \\ 0.27 & 0.35 & 0.38 \end{pmatrix}.$$

Problem no. 6

- This is a M.C. since X_n only depends on X_{n-1} for every n.

$$P = \begin{pmatrix} p_{00} & p_{01} & 0 & \ldots\ldots\ldots\ldots & 0 \\ p_{10} & p_{11} & p_{12} & 0 \ \ldots\ldots\ldots & 0 \\ 0 & p_{21} & p_{22} & p_{32} \quad 0 \quad \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & p_{ii-1} & p_{ii} & p_{ii+1} & 0\cdots \end{pmatrix}$$

- $p_{jj} > 0\,, \quad p_{jj+1} > 0\,, \quad p_{jj-1} > 0 \quad \forall\, j.$
-

$$\begin{aligned} \pi(0) &= 4/79 \\ \pi(1) &= 30/79 \\ \pi(2) &= 45/79 \end{aligned}$$

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$$
\begin{aligned}
E(X) &= 30/79 + 2(45/79) = 120/79 = \underline{1.519} \\
\text{Var}\,(X) &= 30/79 + 2^2(45/79) - (120/79)^2 \\
&= \frac{210}{79} - \left(\frac{120}{79}\right)^2 = \underline{0.3509}
\end{aligned}
$$

Problem no. 7

- The chain is aperiodic since $p_{00} > 0\,,\ p_{11} > 0\,,\ p_{22} > 0$.
- The chain is irreducible since $P^2 > 0$.
- $\pi(0) = 2/5\,,\ \pi(1) = 2/5\,,\ \pi(2) = 1/5$.

Problem no. 8

- $$P = \begin{pmatrix} q_0 & q_1 & q_2 & \cdots & q_k \\ 1 & & & & \\ \vdots & & \bigcirc & & \\ 1 & & & & \end{pmatrix}$$
- $$\pi(0) = 1/(2 - q_0)$$
- $$\pi(i) = q_i/(2 - q_0) \quad i = 1, 2, \ldots, k\ .$$

Problem no. 9

- Rule no. 2 can be used to show that this is a continuous-time Markov chain. We observe that this is actually a birth & death process with birth rate λ_i in state $i \geq 0$ and death rate μ_i in state $i \geq 1$ given by

  $$
  \begin{aligned}
  &\lambda_i = \lambda(3 - i) \quad \text{for} \quad 0 \leq i \leq 3 \\
  &\mu_i = \mu\, i \qquad\quad\ \ \text{for} \quad 0 \leq i \leq 3
  \end{aligned}
  $$

  Its infinitesimal generator Q is given by

  $$
  Q = \begin{pmatrix} -3\lambda & 3\lambda & 0 & 0 \\ \mu & -(2\lambda + \mu) & 2\lambda & 0 \\ 0 & 2\mu & -(\lambda + 2\mu) & \lambda \\ 0 & 0 & 3\mu & -3\mu \end{pmatrix}
  $$

- This C-MC is irreducible. Solving $\pi Q = 0$ and $\pi \cdot \mathbf{1} = 1$ yields

$$
\begin{aligned}
\pi(0) &= 1/(1+3\rho+3\rho^2+\rho^3) = \mathbf{1}/(1+\rho)^3 \\
\pi(0) &= 3\rho\, \pi(0) \\
\pi(2) &= 3\rho^2\, \pi(0) \\
\pi(3) &= \rho^3\, \pi(0)
\end{aligned}
$$

  with $\rho \ : \ = \lambda/\mu$.

  This solution being strictly positive and the C-MC being irreducible (obvious from the shape of Q) we may conclude from Prop. 5 in the Lecture Notes that $\pi(i)$, $i = 0, 1, \ldots, 3$ obtained above are the steady-state probabilities.

  Observe that this result could have also been obtained (better) by applying the result on birth & death processes (Prop. 7 in the Lecture Notes).

- The mean number of busy lines $N_b = \sum_{i=0}^{3} i\pi(i)$ is

$$
\begin{aligned}
N_b &= \frac{3\rho+6\rho^2+3\rho^3}{1+3\rho+3\rho^2+\rho^3} \\
&= \frac{3\rho}{1+\rho}\ .
\end{aligned}
$$