UBINET: Performance Evaluation of Networks

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List of problems No. 1

1. Let X_1 and X_2 be two independent exponentially distributed r.v.'s with parameter $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively.

   Determine the cumulative distribution function of

   - $Z_1 := \min(X_1, X_2)$; (any comment?)
   - $Z_2 := \max(X_1, X_2)$;
   - $Z_3 := X_1 + X_2$.

2. Consider a stream of packets arriving at a communication network in an interval of time of t seconds. We assume that the arrival times of these packets can be modeled as a Poisson process with rate $0.3t$. Compute the probabilities of the following events:

   - Exactly three packets will arrive during a ten-second interval;
   - At most twenty packets will arrive in a period of twenty seconds;
   - The number of packets in an interval of duration five seconds is between three and seven.

3. Let $N, X_1, X_2, \ldots, X_n, \ldots$ be mutually independent r.v.'s, where $N \in \{1, 2, \ldots\}$ and $X_n \geq 0$ for all $n \geq 1$. We assume that $E[N] < \infty$ and that $E[X_n] = E[X_m] := M < \infty$ for all $m, n \geq 1$, $m \neq n$. Define the *random* sum S as $S = \sum_{n=1}^{N} X_n$.

   Determine $E[S]$ as a function of M and $E[N]$.

4. Suppose that whether or not it rains tomorrow depends on the previous weather conditions in the following manner. If it rained today and rained yesterday then it will rain tomorrow with probability 0.8; if it rained yesterday but not today then it will rain tomorrow with probability 0.2, if it rained today but not yesterday, it will rain tomorrow with probability 0.5, and if it rained neither yesterday nor today, it will rain tomorrow with probability 0.1.

   Define the state of the weather, such that its behavior can be described by a discrete-time Markov chain. Describe the meaning of each state and give the transition probability matrix for the chain.

5. Consider a discrete-time Markov chain having the following transition matrix:

$$P = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}.$$

- What are the limiting probabilities $\pi(i)$ for this Markov chain?
- What is the 2-step transition matrix for this Markov chain?

6. Consider the motion of a particle on the space $\{0, 1, 2, \ldots\}$. If the particle is in state $j \geq 0$ at time n $(n \geq 0)$ then at time $n+1$ it will be either in state $j+1$ with the probability $p_{j,j+1}$, in state $j-1$ with the probability $p_{j,j-1}$, or in state j with the probability p_j, where $p_{j,j-1} + p_j + p_{j,j+1} = 1$ for all $j \geq 0$, where by convention $p_{0,-1} = 0$. Let X_n be the location of the particule at time $n \geq 0$.

   - Say why $(X_n,\, n \geq 0)$ is a discrete-state Markov chain and write its transition matrix P;
   - Give sufficient conditions under which this Markov chain is irreducible and aperiodic;
   - We assume that $X_0 \in \{0, 1, 2\}$. Compute the limiting probabilities $\pi(i)$, $i = 0, 1, 2$ as well as $E[X]$ and $\text{var}(X)$ (with $X := \lim_n X_n$) when $p_{0,0} = 0.25$, $p_{0,1} = 0.75$, $p_{1,0} = 0.1$, $p_{1,1} = 0.3$, $p_{1,2} = 0.6$, $p_{2,1} = 0.4$, $p_{2,2} = 0.6$, $p_{2,3} = 0$.

7. Assume that a computer system is in one of 3 states: busy, idle, or undergoing repair, respectively, denoted by states 0, 1, and 2, respectively. Observing its state at 2pm each day, we believe that the system approximately behaves like a homogeneous discrete-time Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{pmatrix}$$

   Prove that the chain is irreducible and aperiodic, and determine the stationary probabilities.

8. Consider a model of a *uniprogrammed* computer system with K I/O devices and a CPU. For the program currently under execution, the system will be in one of the $K+1$ states denoted by $0, 1, \ldots, K$, so that in state 0 the program is using the CPU, and in state k $(1 \leq k \leq K)$ the program is performing an I/O operation on device k. Assume that the request for device k occurs at the end of a CPU burst with probability q_k, independent of the past history of the program. The program will finish execution at the end of a CPU burst with the probability q_0 so that $\sum_{k=0}^{K} q_k = 1$. We assume that the system is saturated so that upon completion of one program, another statistically identical program will enter the system instantaneously. We assume that $0 < q_k < 1$ for $k = 0, 1, \ldots, K$.

   With these assumptions the system can be modeled as an irreducible aperiodic discrete-time Markov chain on the state-space $\{0, 1, \ldots, K\}$.

   - Determine the transition matrix P of this Markov chain;
   - Compute the limiting probabilities $\pi(i)$ of this Markov chain.

9. Three people in an office share 3 telephones lines. We assume that the people make phone calls in a Poisson fashion, each with rate $\lambda > 0$, and that they stay over the phone an exponentially distributed time with the parameter $\mu > 0$. We assume that the 3 Poisson processes are mutually independent, that the call durations are mutually independent and further independent of the Poisson processes.

   Let $X(t)$ be the number of busy lines at time t.

   - Is $(X(t),\, t \geq 0)$ a birth and death process? If yes, determine its birth rates λ_n and its death rates μ_n
   - Is $(X(t),\, t \geq 0)$ a continuous-time Markov chain? If yes, write its infinitesimal generator Q in matrix form;
   - Compute the limiting probabilities $\pi(i) = \lim_{t\to\infty} P(X(t) = i)$;
   - Compute the mean number of busy lines in steady-state.

   [Hint: when $0 \leq j \leq 3$ lines are busy then the time until the next phone call is exponentially distributed with rate $(3-j)\lambda$ because the merging of $3-j$ independent Poisson processes with rate λ is a Poisson process with rate $(3-j)\lambda$.]