Assume that the Markov chain (MC) enters state i at time $t=0$.
Let $S(i)$ be the sojourn time in state i, meaning that at time $S(i)$ the MC will leave state i.
Let us show that $S(i)$ is exponentially distributed with rate $-q_{i,i}$,
namely,

$$P(S(i)<x) = 1 - \exp(q_{i,i}x)$$

or, equivalently,

$$P(S(i)>x) = \exp(q_{i,i}x).$$

Recall that $q_{i,i} := \lim_{h \to 0} (P_{i,i}(h)-1)/h$. (1)

We have

$P(S(i)>x+h) = P(S(i)>x$ and MC does not change of state in $(x,x+h))$

$= P(S(i)>x \mid$ MC does not change of state in $(x,x+h))$.
P(MC does not change of state in $(x,x+h)$)

from (Bayes's formula)

$= P(S(i)>x)$. P(MC does not change of state in $(x,x+h)$) (2)

since in a Markov chain what occurs after time x does not depend on what occured before time x (meaning that the event $\{S(i)>x\}$ is independent of whether of not an event will occur in $(x,x+h)$).

Let us focus on P(MC does not change of state in $(x,x+h)$),
that is, the probability that the chain will stay in state i during $(x,x+h)$.

When $h \to 0$ this probability is given by $P_{i,i}(h)$. We see from (1)
that $P_{i,i}(h) = 1 + h\, q_{i,i} + o(h)$.

Introducing this value in (2) gives

$$P(S(i)>x+h) = P(S(i)>x).(1 + h\, q_{i,i} + o(h))$$

or

$$P(S(i)>x+h) - P(S(i)>x) = P(S_i>x).(h\, q_{i,i} + o(h)).$$

Divide by h and let $h \to 0$ (recall that $o(h)/h \to 0$ as $h \to 0$)
gives the following first order differential equation for $P(S(i)>x)$

$$dP(S(i)>x)/dx = q_{i,i}.P(S(i)>x)$$

whose unique solution such that $P(S(i)>0)=1$ is
$P(S(i)>x) = \exp(q_{i,i}x)$. This proves the result.