

Game Theory: introduction and applications to computer networks

Lecture 1: introduction

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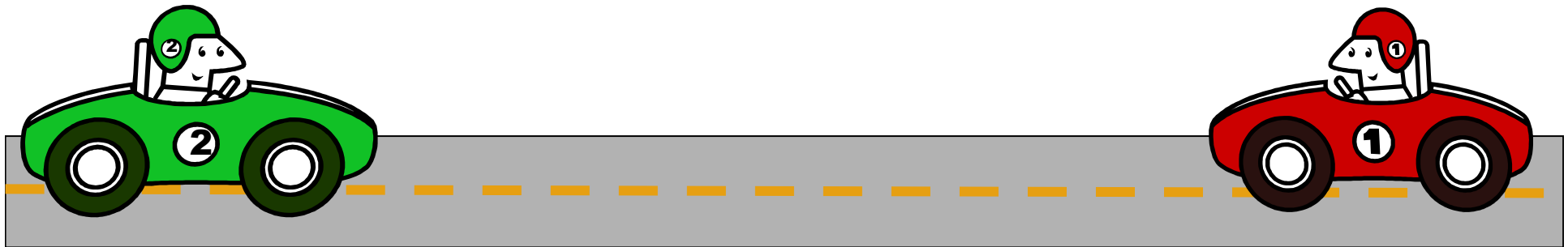
INRIA – EPI Maestro

9 December 2009

Slides are based on a previous course
with D. Figueiredo (UFRJ) and H. Zhang (Suffolk University)

What is Game Theory About?

- Mathematical/Logical analysis of situations of conflict and cooperation



- Game of Chicken
 - driver who steers away loses
 - what should drivers do?
- Goal: to prescribe how rational players should act

What is a Game?

- A Game consists of
 - at least two players
 - a set of strategies for each player
 - a preference relation over possible outcomes
- Player is general entity
 - individual, company, nation, protocol, animal, etc
- Strategies
 - actions which a player chooses to follow
- Outcome
 - determined by mutual choice of strategies
- Preference relation
 - modeled as utility (payoff) over set of outcomes

Short history of GT

- ❑ Forerunners:
 - Waldegrave's first minimax mixed strategy solution to a 2-person game (1713), Cournot's duopoly (1838), Zermelo's theorem on chess (1913), Borel's minimax solution for 2-person games with 3 or 5 strategies (20s)
- ❑ 1928: von Neumann's theorem on two-person zero-sum games
- ❑ 1944: von Neumann and Morgenstern, *Theory of Games and Economic Behaviour*
- ❑ 1950-53: Nash's contributions (Nash equilibrium, bargaining theory)
- ❑ 1952-53: Shapley and Gillies' core (basic concept in cooperative GT)
- ❑ 60s: Aumann's extends cooperative GT to non-transferable utility games
- ❑ 1967-68: Harsanyi's theory of games of incomplete information
- ❑ 1972: Maynard Smith's concept of an Evolutionarily Stable Strategy
- ❑ Nobel prizes in economics
 - 1994 to Nash, Harsanyi and Selten for "their pioneering analysis of equilibria in the theory of non-cooperative games"
 - 2005 to Aumann and Schelling "for having enhanced our understanding of conflict and cooperation through game-theory analysis"
- ❑ Movies:
 - 2001 "A beautiful mind" on John Nash's life
- ❑ See also:
 - www.econ.canterbury.ac.nz/personal_pages/paul_walker/gt/hist.htm

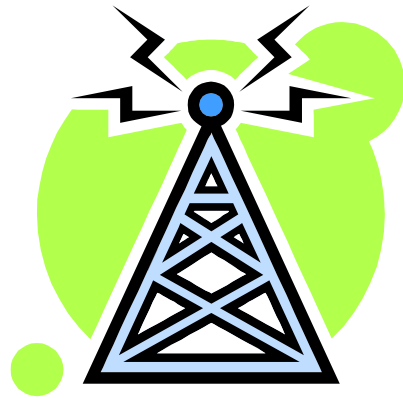
Applications of Game Theory

- Economy
- Politics (vote, coalitions)
- Biology (Darwin's principle, evolutionary GT)
- Anthropology
- War
- Management-labor arbitration
- Philosophy (morality and free will)
- National Football league draft

Applications of Game Theory

- “Recently” applied to computer networks
 - Nagle, RFC 970, 1985
 - “datagram networks as a multi-player game”
 - wider interest starting around 2000
- Which are the strategies available?
 - Network elements follow protocol!!!

Power games



$$SNIR_1 = \frac{H_{1,BS} P_1}{N + H_{2,1} P_2}$$



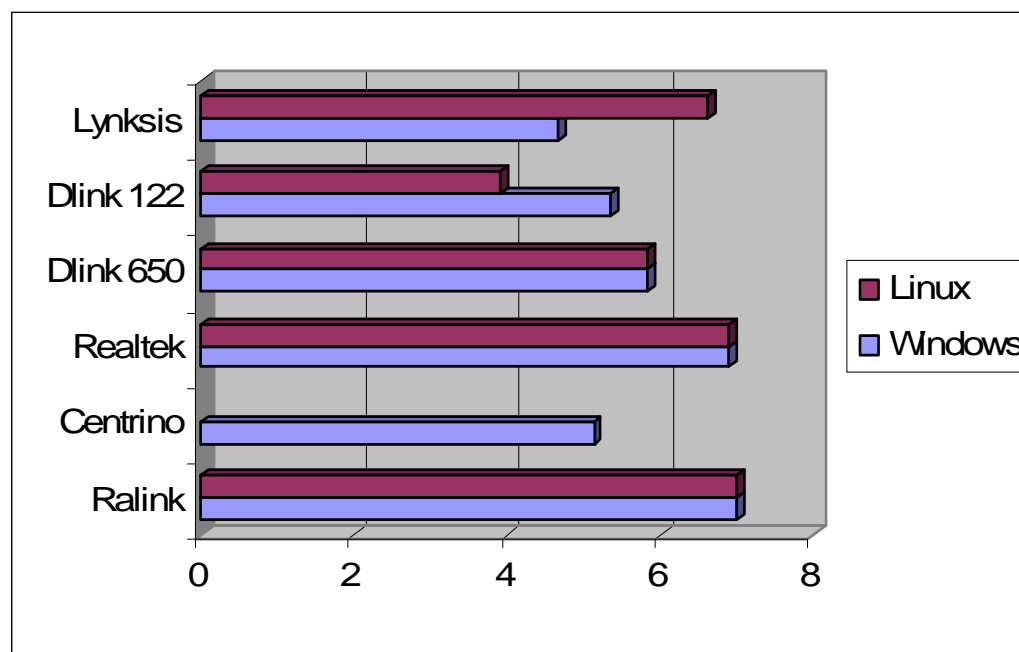
Medium Access Control Games



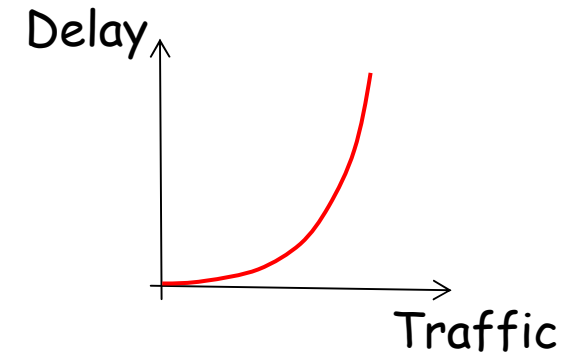
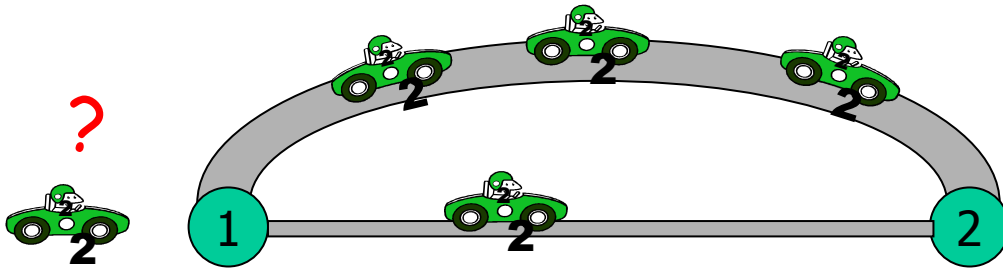
$$Thr_1 = \frac{p_1(1-p_2)P}{(1-p_1)(1-p_2)\sigma + [1-(1-p_1)(1-p_2)]T}$$

Medium Access Control Games

- Despite of the Wi-Fi certification, several cards exhibit very heterogeneous performance, due to arbitrary protocol implementations
 - "Experimental Assessment of the Backoff Behavior of Commercial IEEE 802.11b Network Cards," G Bianchi et al, INFOCOM 2007

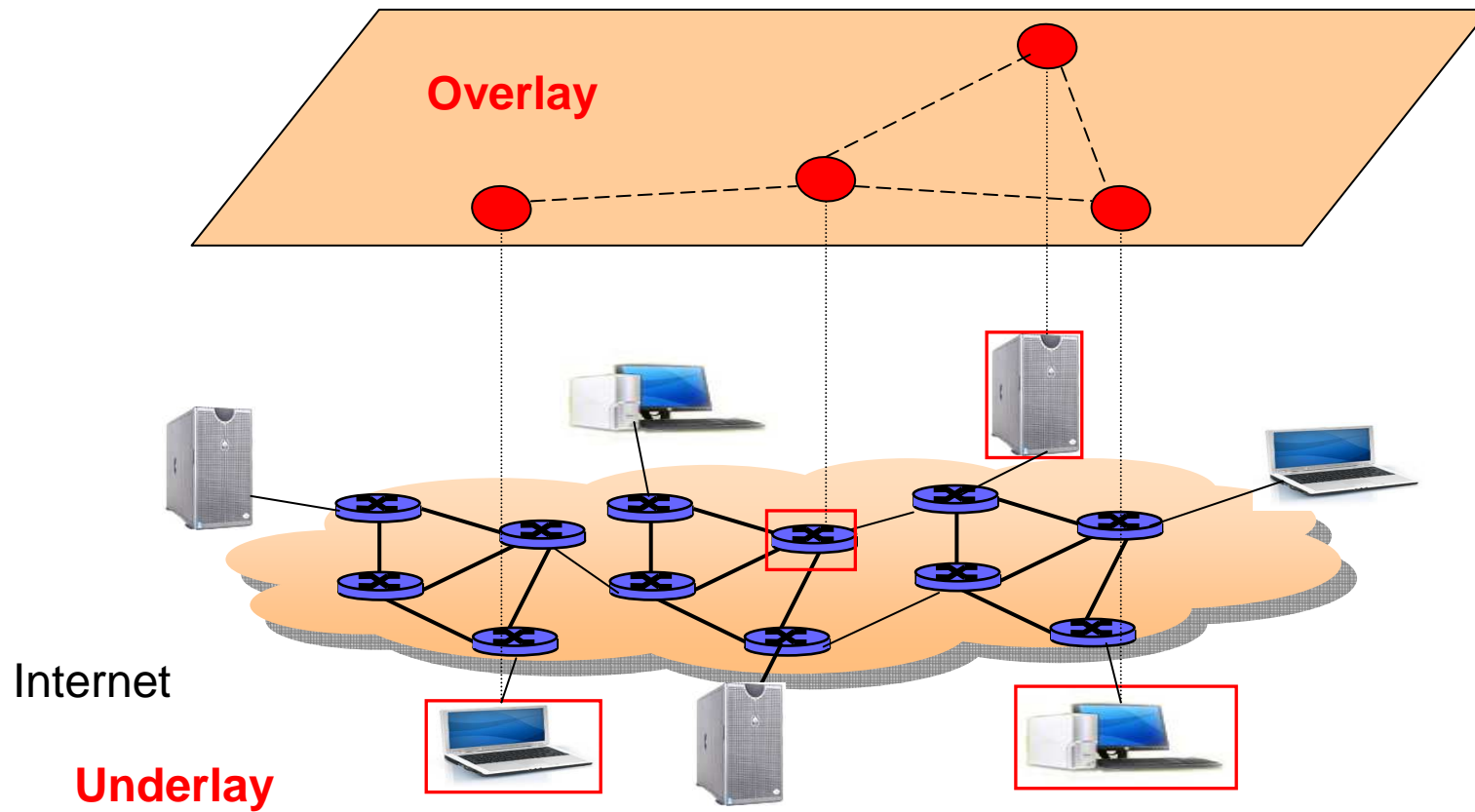


Routing games

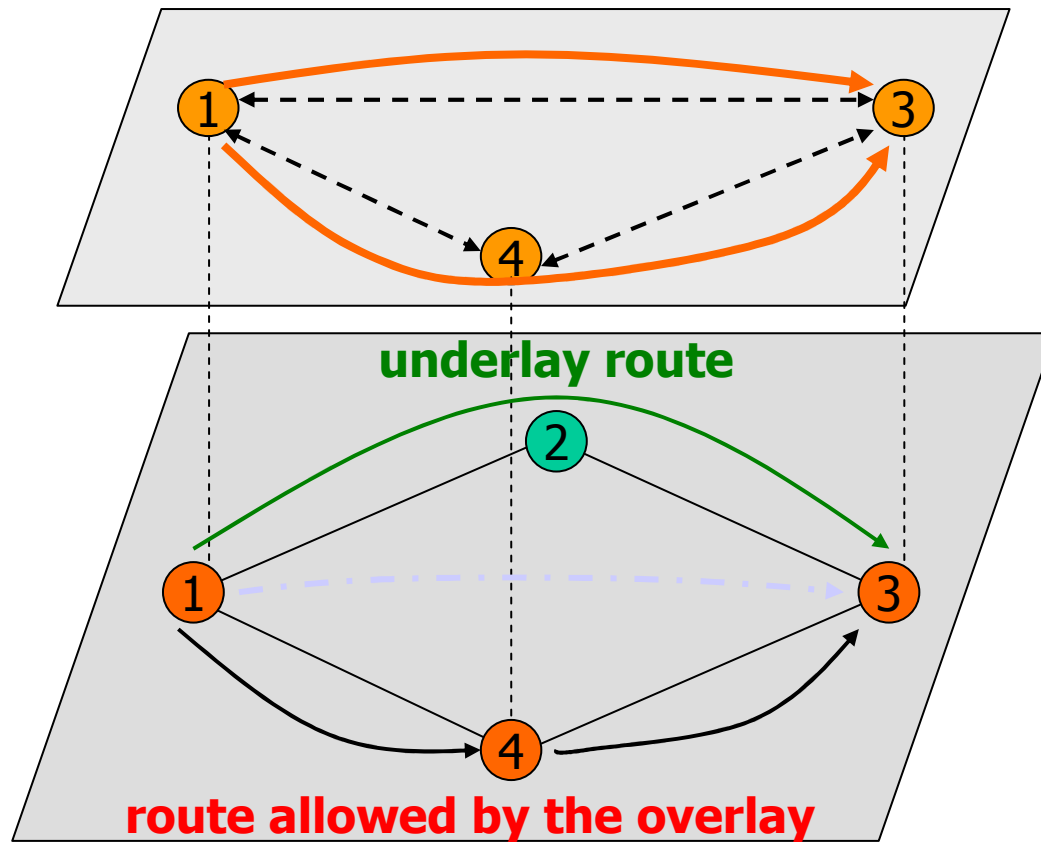


Possible in the Internet?

Overlay networks



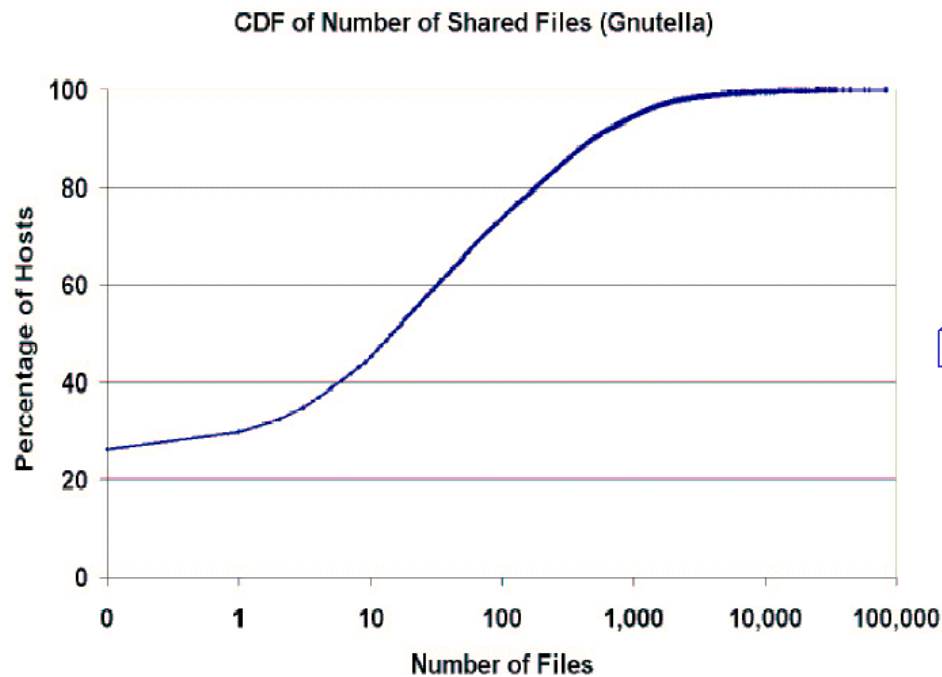
Routing games



An Overlay for routing:
Resilient Overlay Routing

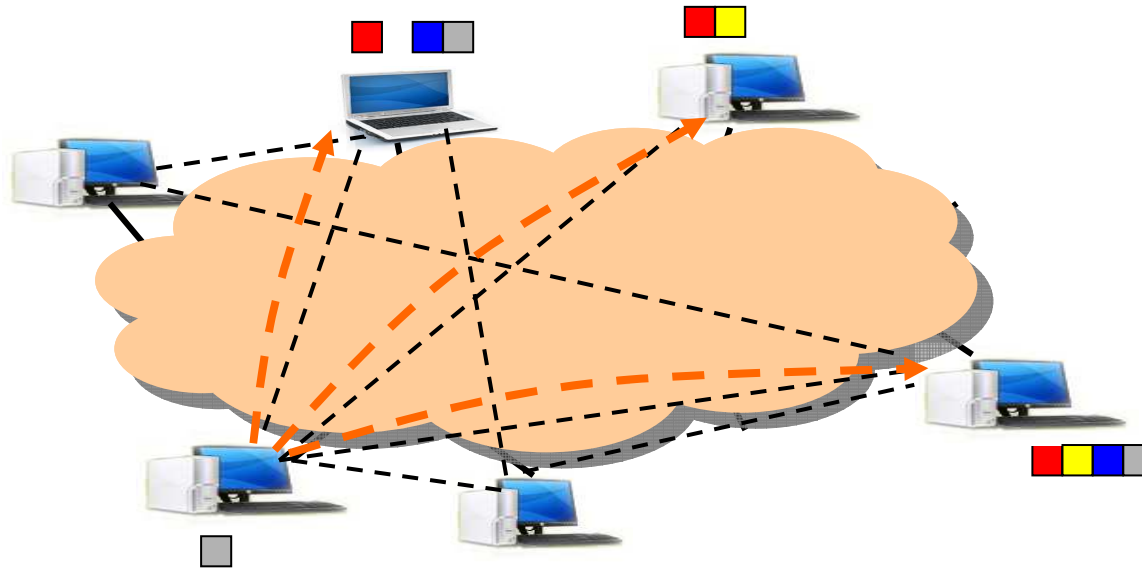
- ❑ Users can ignore ISP choices

Free riders in P2P networks



- ❑ Individuals not willing to pay the cost of a public good, they hope that someone else will bear the cost instead
- ❑ Few servers become the hot spots:
Anonymous?,
Copyright?, Privacy?
Scalability?, Is it P2P?

Connection games in P2P



- Each peer may open multiple TCP connections to increase its downloading rate

Diffusion of BitTorrent variants

- Try to exploit BitTorrent clients weaknesses



BitThief



- Are they really dangerous?
 - Evolutionary game theory says that
Yes they can be

Space for GT in Networks

- ❑ User behaviors (to share or not to share)
 - Client variants
- ❑ Protocols do not specify everything...
 - power level to use
 - number of connections to open
- ❑ ...and/or are not easy to enforce
 - how control a P2P network
 - not-compliant WiFi implementation
- ❑ ...and software easy to modify

Limitations of Game Theory

- ❑ Real-world conflicts are complex
 - models can at best capture important aspects
 - ❑ Players are considered rational
 - determine what is best for them given that others are doing the same
 - Men are not, but computers are more
 - ❑ No unique prescription
 - not clear what players should do
- ❑ But it can provide intuitions, suggestions and partial prescriptions
 - the best mathematical tool we have

Outline

- Following "Straffin, Game Theory and Strategy":
- Two-person zero-sum games
 - Matrix games
 - Pure strategy equilibria (dominance and saddle points), ch 2
 - Mixed strategy equilibria, ch 3
 - Game trees, ch 7
 - About utility, ch 9
- Two-person non-zero-sum games
 - Nash equilibria...
 - ...And its limits (equivalence, interchangeability, Prisoner's dilemma), ch. 11 and 12
 - Strategic games, ch. 14
 - Evolutionary games, ch. 15
- N-persons games

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Lecture 1: two-person zero-sum games

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Matrix Game (Normal form)

Strategy set for Player 1

Player 1, Rose

Player 2, Colin

Strategy set for Player 2

		Player 2, Colin		
		A	B	C
Player 1, Rose	A	(2, 2)	(0, 0)	(-2, -1)
	B	(-5, 1)	(3, 4)	(3, -1)

Payoff to Player 1

Payoff to Player 2

- Simultaneous play

- players analyze the game and then write their strategy on a piece of paper

More Formal Game Definition

- Normal form (strategic) game
 - a finite set N of players
 - a set strategies S_i for each player $i \in N$
 - payoff function $u_i(s)$ for each player $i \in N$
 - where $s \in S = \prod_{j \in N} S_j$ is an outcome
 - sometimes also $u_i(A, B, \dots)$ $A \in S_1, B \in S_2, \dots$
 - $u_i : S \rightarrow \mathfrak{R}$

Two-person Zero-sum Games

- One of the first games studied
 - most well understood type of game
- Players interest are strictly opposed
 - what one player gains the other loses
 - game matrix has single entry (gain to player 1)
- A "strong" solution concept

Let's play!

		Colin			
		A	B	C	D
Rose	A	12	-1	1	0
	B	5	1	7	-20
	C	3	2	4	3
	D	-16	0	0	16

- Divide in pairs, assign roles (Rose/Colin) and play 20 times
- Log how many times you have played each strategy and how much you have won

Analyzing the Game

		Colin			
		A	B	C	D
Rose	A	12	-1	1	0
	B	5	1	7	-20
	C	3	2	4	3
	D	-16	0	0	16

dominated strategy (dominated by B)

Dominance

- Strategy S (*weakly*) dominates a strategy T if every possible outcome when S is chosen is at least as good as corresponding outcome in T , and one is strictly better
 - S strictly dominates T if every possible outcome when S is chosen is strictly better than corresponding outcome in T
- Dominance Principle
 - rational players never choose dominated strategies
- Higher Order Dominance Principle
 - iteratively remove dominated strategies

Higher order dominance may be enough

		Colin			
		A	B	C	D
Rose	A	12	-1	1	0
	B	3	1	4	-18
	C	5	2	4	3
	D	-16	0	5	-1

Higher order dominance may be enough

GT prescribes:

Rose C - Colin B

		Colin			
		A	B	C	D
Rose	A	12	-1	1	0
	B	3	1	4	-18
	C	5	2	4	3
	D	-16	0	5	-1

(Weakly) Dominated by C

A priori D is not dominated by C

Strictly dominated by B

... but not in the first game

		Colin			
		A	B	C	D
Rose	A	12	-1	1	0
	B	5	1	7	-20
	C	3	2	4	3
	D	-16	0	0	16

dominated strategy
(dominated by B)

Analyzing the Reduced Game: Movement Diagram

		Colin		
		A	B	D
Rose	A	12	-1	0
	B	5	1	-20
	C	3	2	3
	D	-16	0	16

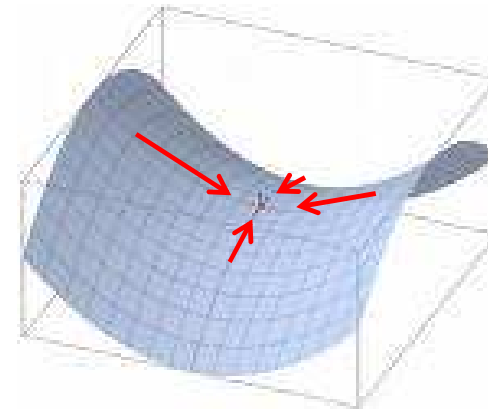
Outcome (C, B) is
"stable"

- *saddle point of game*
- *mutual best responses*

Saddle Points

- An outcome (x,y) is a *saddle point* if the corresponding entry $u(x,y)$ is both less than or equal to any value in its row and greater than or equal to any value in its column
 - $u(x,y) \leq u(x,w)$ for all w in $S_2 = S_{\text{Colin}}$
 - $u(x,y) \geq u(v,y)$ for all v in $S_1 = S_{\text{Rose}}$

	A	B	D
A	12	-1	0
B	5	1	-20
C	3	2	3
D	-16	0	16



Saddle Points Principle

- Players should choose outcomes that are saddle points of the game
 - Because it is an equilibrium...
 - ... but not only

Saddle Points main theorem

- The game has a saddle point iff

$$\max_v \min_w u(v,w) = \min_w \max_v u(v,w)$$

		Colin			\min_w
		A	B	D	
Rose	A	12	-1	0	-1
	B	5	1	-20	-20
	C	3	2	3	2
	D	-16	0	16	-16
	\max_v	12	2	16	

- Rose $C \in \operatorname{argmax} \min_w u(v,w)$
most cautious strategy for Rose: it secures the maximum worst case gain independently from Colin's action
(the game *maximin value*)

- Colin $B \in \operatorname{argmin} \max_v u(v,w)$
most cautious strategy for Colin: it secures the minimum worst case loss
(the game *minimax value*)

Saddle Points main theorem

- Another formulation:
 - The game has a saddle point iff
maximin = minimax,
- This value is called the **value of the game**

Saddle Points main theorem

- The game has a saddle point iff
$$\max_v \min_w u(v,w) = \min_w \max_v u(v,w)$$

N.C.

Two preliminary remarks

1. It holds (always)

$$\max_v \min_w u(v,w) \leq \min_w \max_v u(v,w)$$

because $\min_w u(v,w) \leq u(v,w) \leq \max_v u(v,w)$ for all v and w

2. By definition, if (x,y) is a saddle point

- $u(x,y) \leq u(x,w)$ for all w in S_{Colin}
 - i.e. $u(x,y) = \min_w u(x,w)$
- $u(x,y) \geq u(v,y)$ for all v in S_{Rose}
 - i.e. $u(x,y) = \max_v u(v,y)$

Saddle Points main theorem

□ The game has a saddle point iff

$$\max_v \min_w u(v,w) = \min_w \max_v u(v,w)$$

1. $\max_v \min_w u(v,w) \leq \min_w \max_v u(v,w)$

2. if (x,y) is a saddle point

○ $u(x,y) = \min_w u(x,w), \quad u(x,y) = \max_v u(v,y)$

N.C.

$$u(x,y) = \min_w u(x,w) \leq \max_v \min_w u(v,w) \leq \min_w \max_v u(v,w) \leq \max_v u(v,y) = u(x,y)$$

Saddle Points main theorem

- The game has a saddle point iff $\max_v \min_w u(v,w) = \min_w \max_v u(v,w)$

S.C.

x in $\operatorname{argmax}_v \min_w u(v,w)$

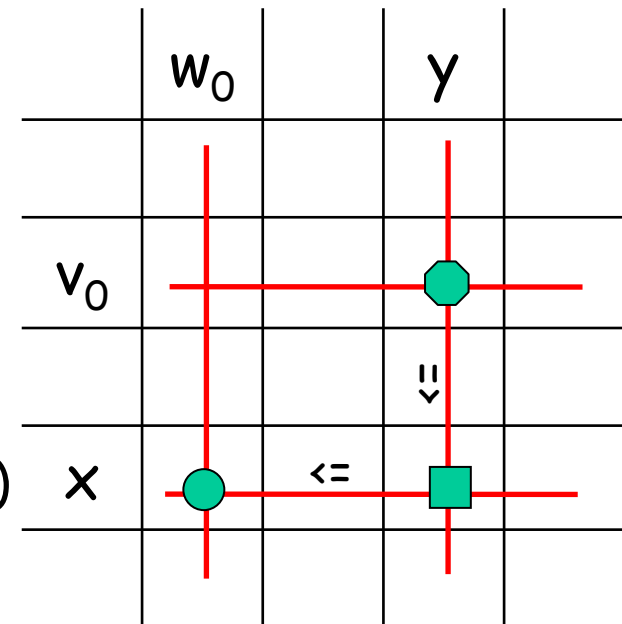
y in $\operatorname{argmin}_w \max_v u(v,w)$

We prove that (x,y) is a saddle-point

w_0 in $\operatorname{argmin}_w u(x,w)$ ($\max_v \min_w u(v,w) = u(x,w_0)$)

v_0 in $\operatorname{argmax}_v u(v,y)$ ($\min_w \max_v u(v,w) = u(v_0,y)$)

$u(x,w_0) = \min_w u(x,w) \leq u(x,y) \leq \max_v u(v,y) = u(v_0,y)$



Note that $u(x,y) = \max_v \min_w u(v,w)$

Saddle Points main theorem

- The game has a saddle point iff

$$\max_v \min_w u(v,w) = \min_w \max_v u(v,w)$$

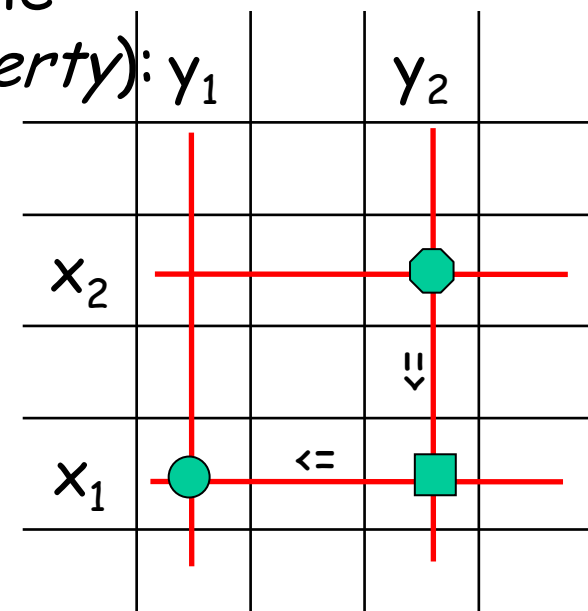
		Colin			min _w
		A	B	D	
Rose	A	12	-1	0	-1
	B	5	1	-20	-20
	C	-3	2	3	2
	D	-16	0	16	-16
max _v		12	2	16	

This result provides also another way to find saddle points

Properties

- Given two saddle points (x_1, y_1) and (x_2, y_2) ,
 - they have the same payoff (*equivalence property*):
 - it follows from previous proof:
$$u(x_1, y_1) = \max_v \min_w u(v, w) = u(x_2, y_2)$$
 - (x_1, y_2) and (x_2, y_1) are also saddle points (*interchangeability property*):
 - as in previous proof

They make saddle point
a very nice solution!



What is left?

- There are games with no saddle-point!
- An example?

	R	P	S	min
R	0	-1	1	-1
P	1	0	-1	-1
S	-1	1	0	-1
max	1	1	1	

minimax



maximin

maximin <> minimax

What is left?

- There are games with no saddle-point!
- An example? An even simpler one

	A	B	min	
A	2	0	0	maximin
B	-5	3	-5	
max	2	3		minimax

Some practice: find all the saddle points

	A	B	C	D
A	3	2	4	2
B	2	1	3	0
C	2	2	2	2

	A	B	C
A	-2	0	4
B	2	1	3
C	3	-1	-2

	A	B	C
A	4	3	8
B	9	5	1
C	2	7	6

Games with no saddle points

		Colin	
		A	B
Rose	A	2 → 0	
	B	← -5 ← 3	

- What should players do?
 - resort to randomness to select strategies

Mixed Strategies

- Each player associates a probability distribution over its set of strategies
- Expected value principle: maximize the expected payoff

		Colin	
		1/3	2/3
Rose	A	2	0
	B	-5	3

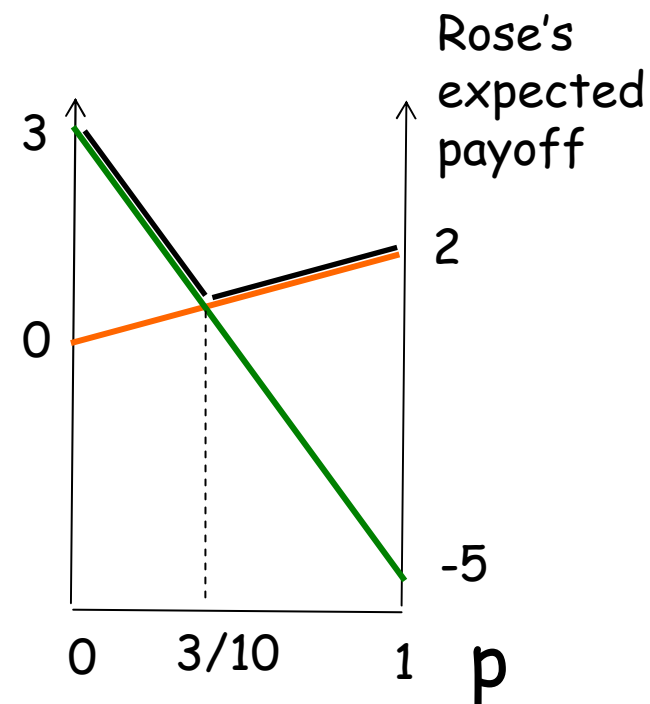
Rose's expected payoff when playing A = $1/3 \cdot 2 + 2/3 \cdot 0 = 2/3$

Rose's expected payoff when playing B = $1/3 \cdot -5 + 2/3 \cdot 3 = 1/3$

- How should Colin choose its prob. distribution?

2x2 game

		Colin	
		p	1-p
Rose	A	2	0
	B	-5	3



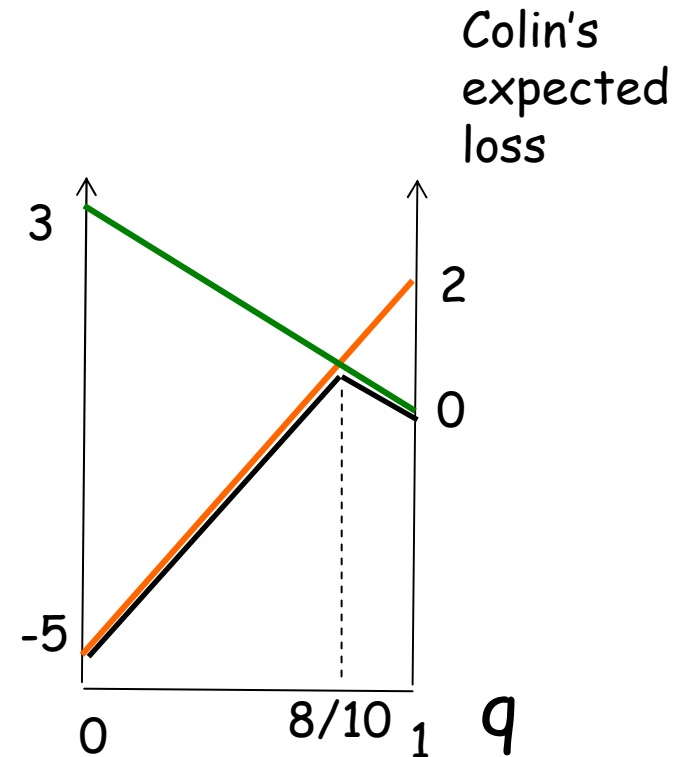
Rose's exp. gain when playing A = $2p + (1-p)*0 = 2p$

Rose's exp. gain when playing B = $-5*p + (1-p)*3 = 3-8p$

- How should Colin choose its prob. distribution?
 - Rose cannot take advantage of $p=3/10$
 - for $p=3/10$ Colin guarantees a loss of $3/5$, what about Rose's?

2x2 game

		Colin		
		A	B	
Rose	q	A	2	0
	1-q	B	-5	3



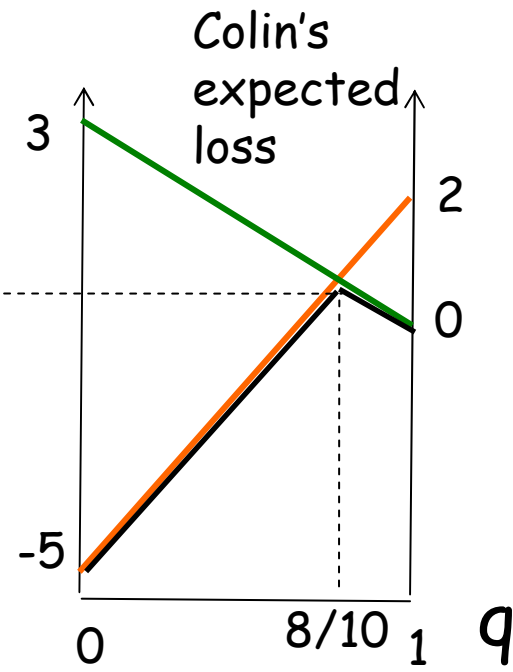
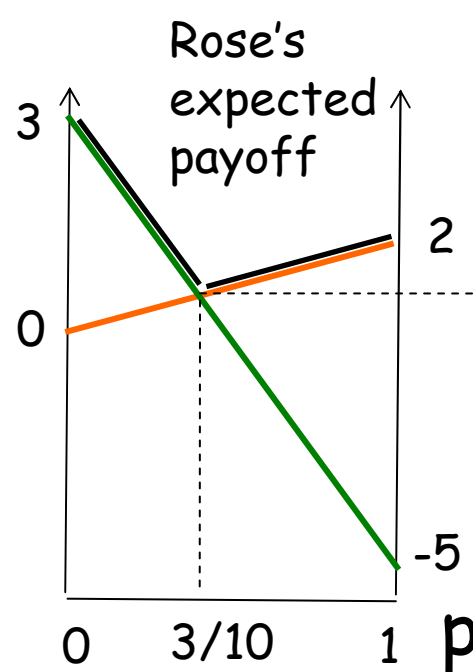
Colin's exp. loss when playing A = $2q - 5*(1-q) = 7q - 5$

Colin's exp. loss when playing B = $0*q + 3*(1-q) = 3 - 3q$

- How should Rose choose its prob. distribution?
 - Colin cannot take advantage of $q = 8/10$
 - for $q = 8/10$ Rose guarantees a gain of?

2x2 game

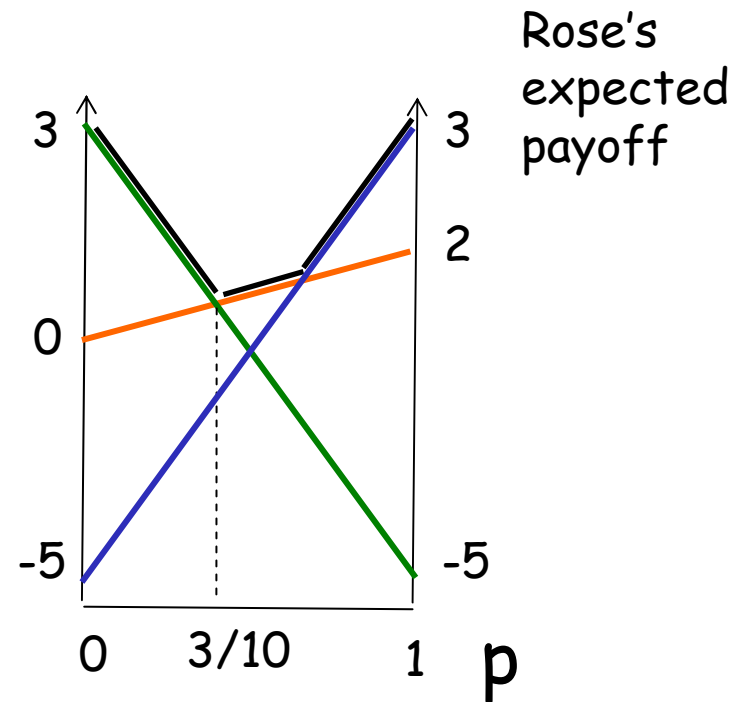
		Colin	
		p	1-p
Rose	q	A	B
	1-q	A	B
		2	0
		-5	3



- Rose playing the mixed strategy $(8/10, 2/10)$ and Colin playing the mixed strategy $(3/10, 7/10)$ is the equilibrium of the game
 - No player has any incentives to change, because any other choice would allow the opponent to gain more
 - Rose gain $3/5$ and Colin loses $3/5$

mx2 game

		Colin	
		p	1-p
Rose	X	A	B
	Y	2	0
	1-X-Y	-5	3



- By playing $p=3/10$, Colin guarantees max exp. loss = $3/5$
 - it loses $3/5$ if Rose plays A or B, it wins $13/5$ if Rose plays C
- Rose should not play strategy C

Minimax Theorem

- Every two-person zero-sum game has a solution, i.e., there is a unique value v (*value of the game*) and there are optimal (pure or mixed) strategies such that
 - Rose's optimal strategy guarantees to her a payoff $\geq v$ (no matter what Colin does)
 - Colin's optimal strategies guarantees to him a payoff $\leq v$ (no matter what Rose does)
- This solution can always be found as the solution of a $k \times k$ subgame
- Proved by John von Neumann in 1928!
 - birth of game theory...

How to solve $m \times n$ matrix games

1. Eliminate dominated strategies
2. Look for saddle points (solution of 1×1 games), if found stop
3. Look for a solution of all the $h \times h$ games, with $h = \min\{m, n\}$, if found stop
4. Look for a solution of all the $(h-1) \times (h-1)$ games, if found stop
5. ...
- $h+1$. Look for a solution of all the 2×2 games, if found stop

How to solve mxm games

- if all the strategies are used at the equilibrium, the probability vector is such to make equivalent for the opponent all its strategies
 - a linear system with $m-1$ equations and $m-1$ variables
 - if it has no solution, then we need to look for smaller subgames

		Colin			
		A	B	C	
Rose	x	A	2	0	1
	y	B	-5	3	-2
	$1-x-y$	C	3	-5	3

Example:

- $2x-5y+3(1-x-y)=0x+3y-5(1-x-y)$
- $2x-5y+3(1-x-y)=1x-2y+3(1-x-y)$

How to solve 2x2 games

- If the game has no saddle point
 - calculate the absolute difference of the payoffs achievable with a strategy
 - invert them
 - normalize the values so that they become probabilities

		Colin	
		p	1-p
Rose	q	A	0
	1-q	B	3

→ $|2-0|=2$ → 8

→ $|-5-3|=8$ → 2

↗ 8/10

↘ 2/10

Utility:

where do numbers come from?

- Only in some (simple) cases a natural quantitative metric
- Utility theory: how to assign numbers to payoffs so that they reflect player's preferences
- For saddle points: only ordinal utilities are needed
 - consistency conditions for player's preferences: player has to be able to rank outcomes
- in general cardinal utilities are needed
 - ratio of differences is meaningful
 - affine transformations do not change the game
 - summing a constant value to all the payoffs
 - multiplying all the payoffs for the same value

How to determine cardinal utilities (Von Neumann and Morgenstern)

- ❑ let player rank outcomes, e.g. $a < b < c < d$
- ❑ assign arbitrary numerical utilities to the least and the most preferred, e.g. $a=0$, $d=100$
- ❑ determine the utilities of the other outcomes by asking the player to compare *lotteries*
 - "do you prefer b to a lottery where you can win d with probability p and a with probability $1-p$?"
 - when the equivalent lott. is determined (with prob. p_e) assign to b the lottery expected value, i.e. $100 * p_e$
- ❑ more sophisticated consistency conditions are required

Utilities: remarks on zero-sum games

- For the game to be zero-sum the other player has to express exactly reverse preferences
- but cardinal utilities are to some extent arbitrary
 - payoff affine transformations do not change the result => "masked" zero-sum games
 - how to recognize them
 - it exists $i, j, h, k \mid i \cdot u_R + j + h \cdot u_C + k = 0, \Rightarrow (u_R, u_C)$ lie on a segment

		Colin		$\xrightarrow{u'_C = 3 \cdot u_C + 6}$	Colin		
		A	B		A	B	
Rose	A	(2, -2)	(0, 0)	Rose	A	(2, 0)	(0, 6)
	B	(-5, 5)	(3, -3)		B	(-5, 21)	(3, -3)

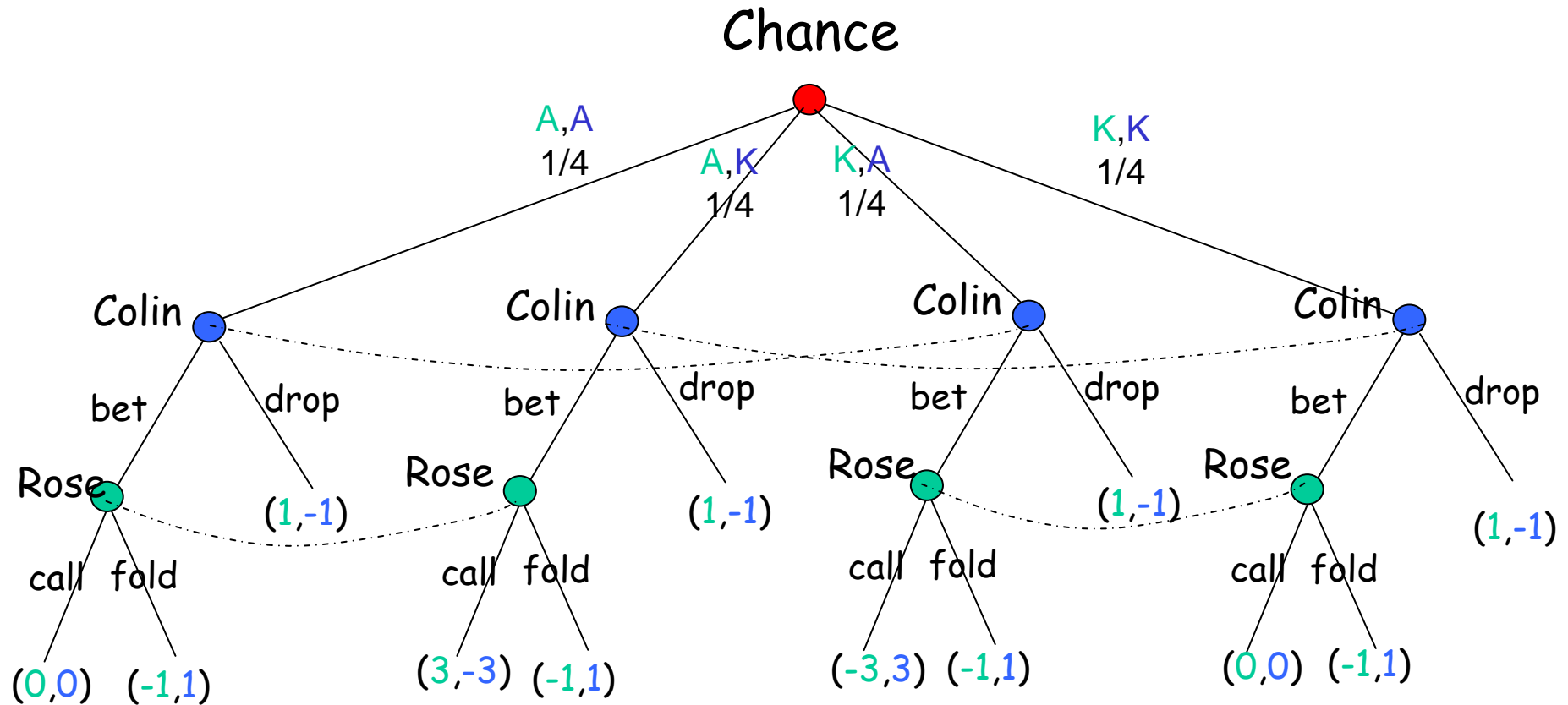
Game Trees (Extensive form)

- Sequential play
 - players take turns in making choices
 - previous choices may be available to players
- Game represented as a tree
 - each non-leaf node represents a decision point for some player
 - edges represent available choices

Game Trees: simplified poker

- ❑ Rose and Colin put 1\$ each in the pot and take a card (Ace or King)
- ❑ Colin may bet other 2\$ or drop
- ❑ If Colin bets
 - Rose can put other 2\$ and call (and the highest card wins)
 - or can fold (and Colin takes the money)
- ❑ If Colin drops
 - Rose takes all the money in the pot

Tree of the simplified poker



- Arc joins states of a player in the same *information set*.
- when playing the player cannot distinguish these states
 - the known sequence of past events is the same
 - the set of future actions is the same

Game trees: more formal definition

1. each node is labeled by the player (including Chance) who makes a choice at that node
2. each branch leading by a node corresponds to a possible choice of the player at the node
3. each branch corresponding to a choice made by Chance is labeled with the corresponding probability
4. each leaf is labeled by players payoffs
5. nodes of each player are partitioned in information sets

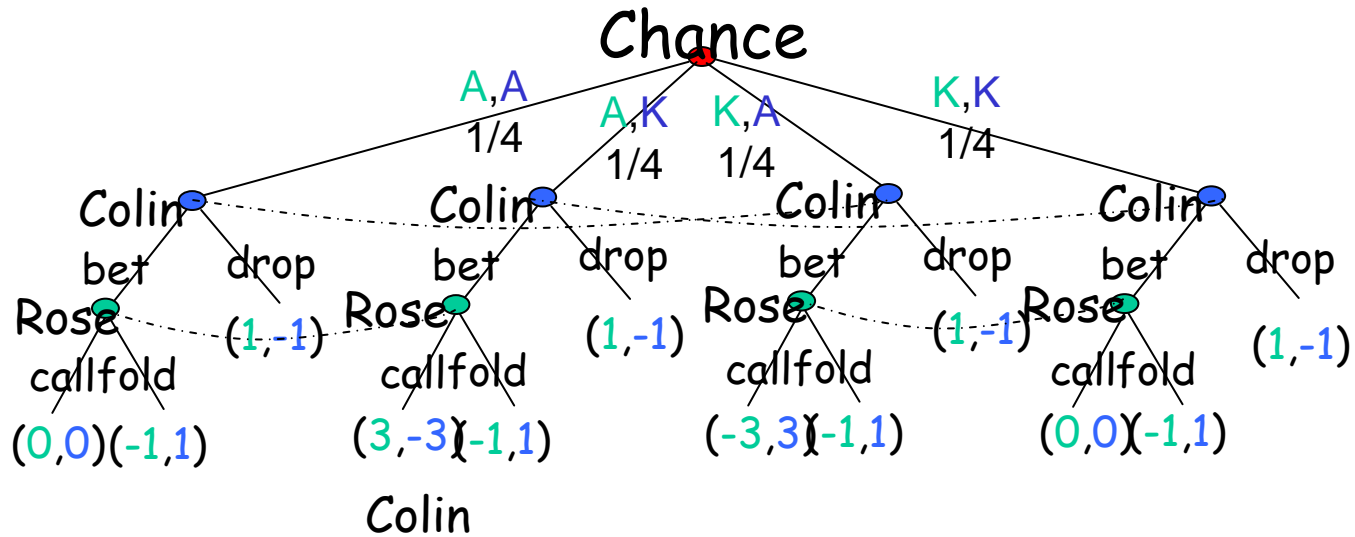
Game trees and matrix games

- ❑ Each game tree can be converted in a matrix game!
- ❑ Connecting idea: strategy in game tree
 - it specifies a priori all the choices of the player in each situation
 - only need to specify for each information set
 - e.g. in simplified poker
 - for Colin 4 possible strategies
 - “always bet” (bb), “bet only if ace” (bd), “bet only if king” (db), “always drop” (dd)
 - for Rose 4 possible strategies
 - “always call” (cc), “call only if ace” (cf), “call only if king” (fc), “always fold” (ff)

Game trees and matrix games

- ❑ Each game tree can be converted in a matrix game!
- ❑ Once identified the strategies of every player...
- ❑ ...use the expected payoffs of the game tree as payoffs of the matrix game

Game trees and matrix games



		Colin			
		bb	bd	db	dd
Rose	cc	0	$-1/4$	$5/4$	1
	cf	$1/4$	$1/4$	1	1
	fc	$-5/4$	$-1/2$	$1/4$	1
	ff	-1	0	0	1

Study this game

Game trees and matrix games

- ❑ Each game tree can be converted in a matrix game!
- ❑ Problem: this approach does not scale with the size of the tree
 - exponential growth in the number of strategies
 - consider how many strategies are available in chess to White and to Black for their respective first move
- ❑ Try to study directly the game tree

Game trees with perfect information

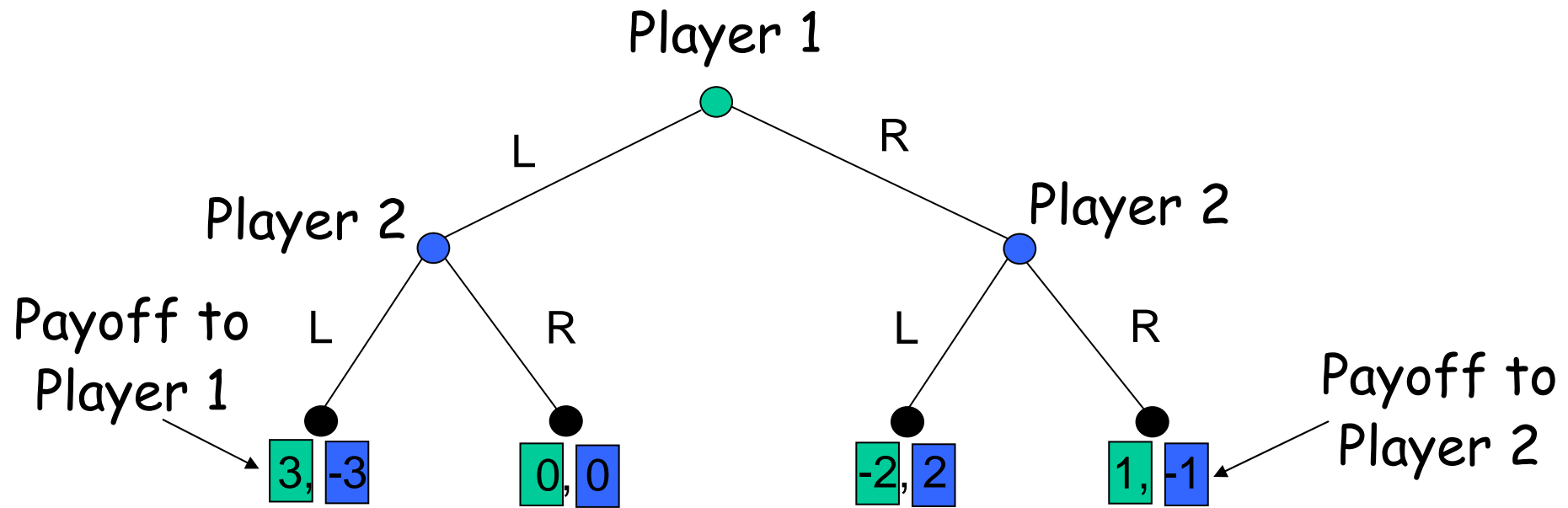
□ Definition

1. no nodes are labeled by *Chance*
2. all information sets consist of a single node

□ Test: which among the following is a game with perfect information and why?

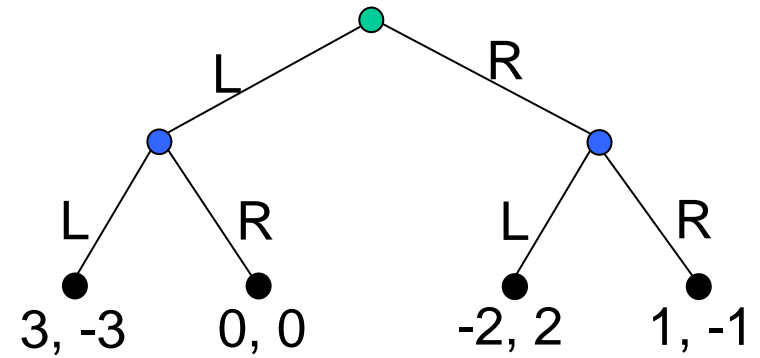
- poker
- tic, tac, toe
- rock, scissor, paper
 - honestly and dishonestly played...
- chess
- guess the number

Perfect information: an example



- Strategy sets
 - for Player 1: $\{L, R\}$
 - for Player 2: $\{LL, LR, RL, RR\}$
- Convert it to a matrix game and solve it

Converting to Matrix Game

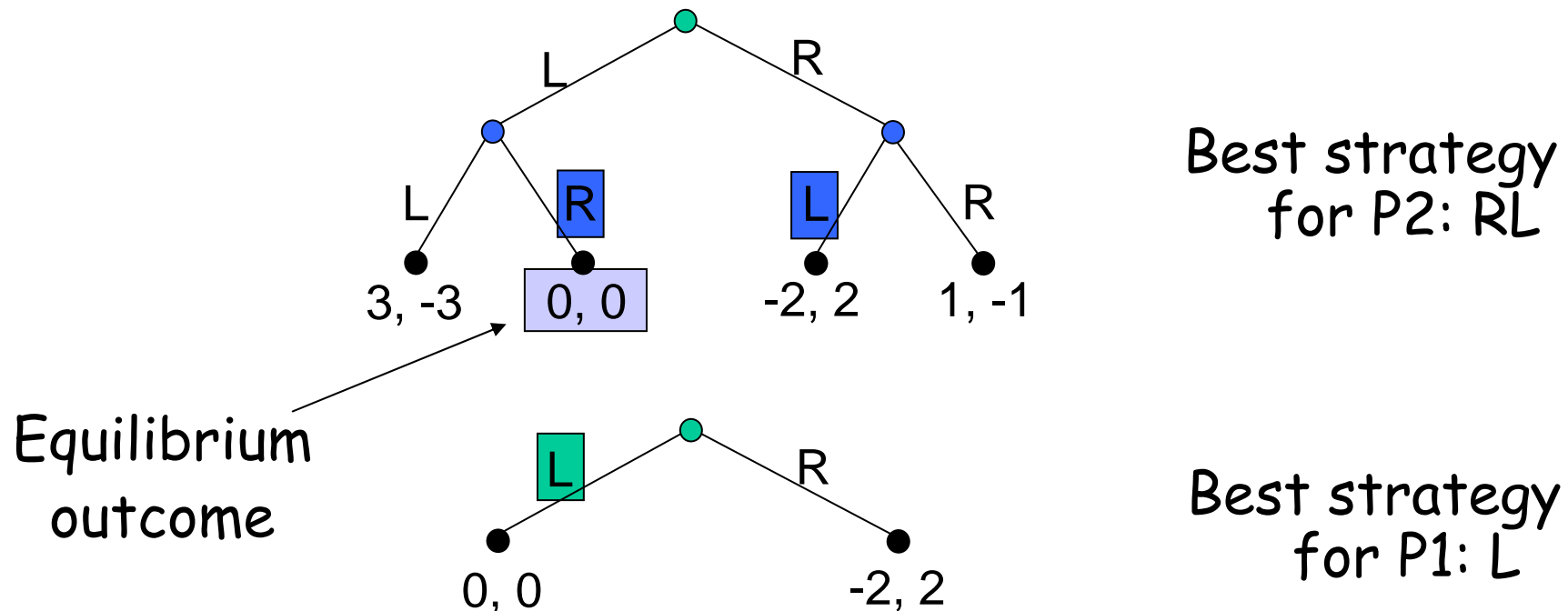


Player 2

	LL	LR	RL	RR
Player 1 L	3	3	0	0
Player 1 R	-2	1	-2	1

Solving the game by backward induction

- Starting from terminal nodes
 - move up game tree making best choice



- Saddle point:
P1 chooses L, P2 chooses RL

Kuhn's Theorem

- Backward induction always leads to saddle point (on games with perfect information)
 - game value at equilibrium is unique (for zero-sum)

- Consequences for chess?
 - at the saddle point
 - or White wins, value = 1 -> White has winning strategy no matter what Black does
 - or Black wins, value = -1 -> Black has winning strategy, no matter what White does
 - or they draw, value = 0 -> Both White and Black have a strategy guaranteeing at least drawing

Chess is a simple game! (Zermelo 1913)