Game Theory: introduction and applications to computer networks

Lecture 3: two-person non zero-sum games

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Slides are based on a previous course

with D. Figueiredo (UFRJ) and H. Zhang (Suffolk University)

Outline

- Two-person zero-sum games
  - Matrix games
    - Pure strategy equilibria (dominance and saddle points), ch 2
    - Mixed strategy equilibria, ch 3
  - Game trees, ch 7
  - About utility, ch 9
- Two-person non-zero-sum games
  - Nash equilibria…
    - …And its limits (equivalence, interchangeability, Prisoner's dilemma), ch. 11 and 12
  - Strategic games, ch. 14
  - Subgame Perfect Nash Equilibria (not in the book)
  - Repeated Games, partially in ch. 12
  - Evolutionary games, ch. 15
- N-persons games

Strategic moves: preliminary

- Is it better to play first or after?
  1. what about zero-sum games with saddle points?
  2. and about zero-sum games without saddle points?
  - Answers: 1. no difference, 2. the first player makes worse

Strategic moves: preliminary

- Is it better to play first or after?
  - and for non zero-sum games?

Colin (columns), Rose (rows)

	swerve	stay
swerve	0, 0	-1, 5
stay	5, -1	-10, -10

Rose makes better

Colin (columns), Rose (rows)

	A	B
A	2, 3	4, 2
B	1, 0	3, 5

Both players would like Rose to play first!

Colin (columns), Rose (rows)

	A	B
A	1, 2	2, 0
B	2, 0	0, 2

Rose makes worse

Strategic moves

- Even if both players play at the same time, similar effects can be obtained if players can communicate
  - commitment, Rose: “I will always play X”
  - how to convince Colin
    - communicate and block any form of communications
    - reduce its own payoffs (public commitment, contract,…)
    - gain credibility in a repeated game

Colin

Rose	A	B
A	2, 3	4, 2
B	1, 0	3, 5

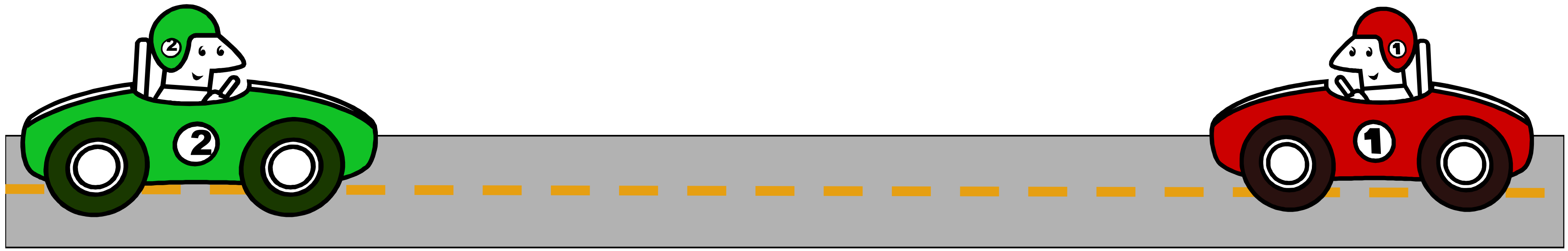
Colin

Rose	A	B
A	0, 3	2, 2
B	1, 0	3, 5

Strategic moves

- Even if both players play at the same time, similar effects can be obtained if players can communicate
  - commitment, Rose: "I will always play X"
- 2 other basic forms of communications in sequential games
  - threat, Rose "If you play X, I will play Y"
    - Y is harmful to Colin, but also to Rose!
  - promise: "If you play X, I will play Y"
    - Y is beneficial to Colin, but harmful to Rose.

Possible conflicting commitments



- Game of Chicken (aka. Hawk-Dove Game)
  - driver who swerves looses

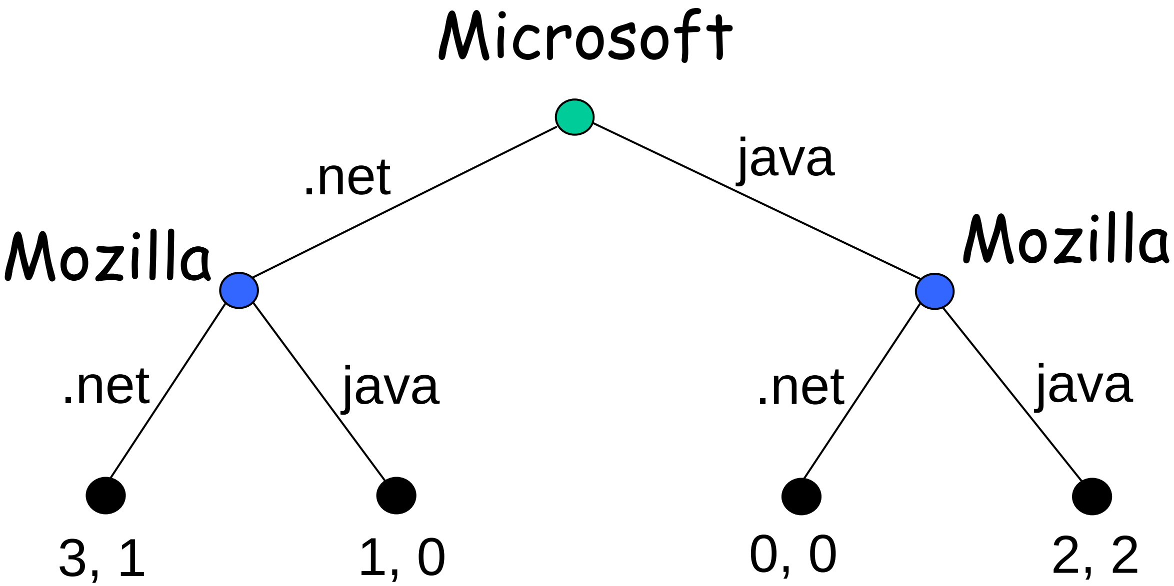
Driver 1 \ Driver 2	swerve	stay
swerve	0, 0	-1, 5
stay	5, -1	-10, -10

Drivers want to do opposite of one another

Will prior communication help?

Game Trees Revisited

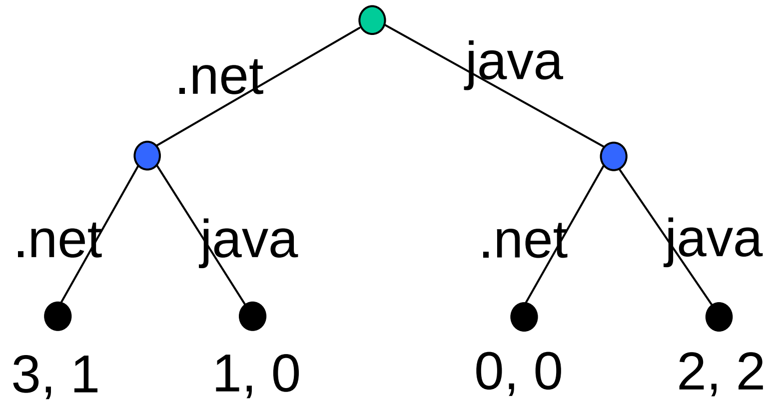
- Microsoft and Mozilla are deciding on adopting new browser technology (.net or java)
  - Microsoft moves first, then Mozilla makes its move



- Non-zero sum game
  - what are the NEs?
  - remember: a (pure) strategy has to specify the action at each information set

NE and Threats

- Convert the game to normal form



Mozilla

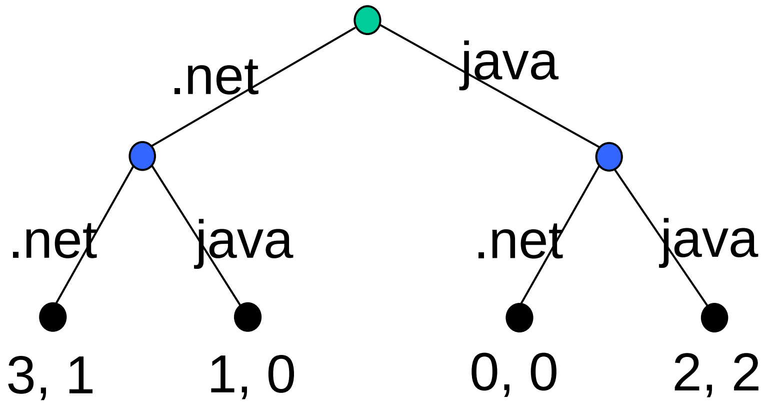
Microsoft		NN	NJ	JN	JJ
	.net	$\boxed{3,1}$	$\boxed{3,1}$	1,0	1,0
	java	0,0	2,2	0,0	$\boxed{2,2}$

NE

- Mozilla's JJ is a threat to Microsoft
  - I will play Java, no matter what you do
  - harmful to Microsoft, but also to Mozilla if Microsoft plays .net

NE and Threats

- Convert the game to normal form



Mozilla

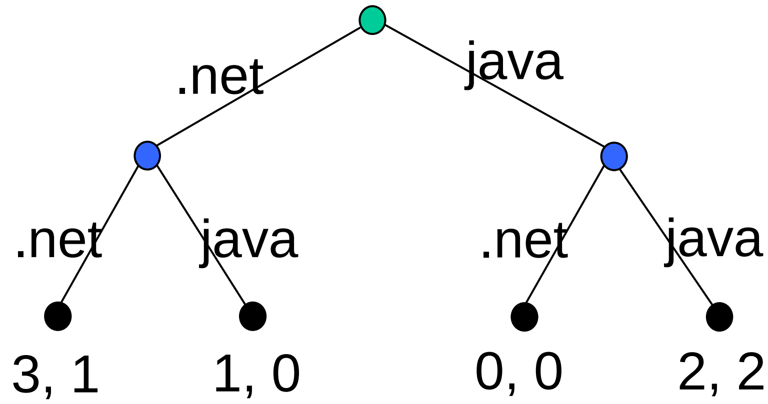
Microsoft		NN	NJ	JN	JJ
	.net	3,-1	3,-1	1,0	1,0
	java	0,-2	2,2	0,-2	2,2

NE

- Mozilla's JJ is a threat to Microsoft
- Mozilla may declare that it will never adopt .net (loss of image when adopting .net equal to -2)

NE and Incredible Threats

- Convert the game to normal form



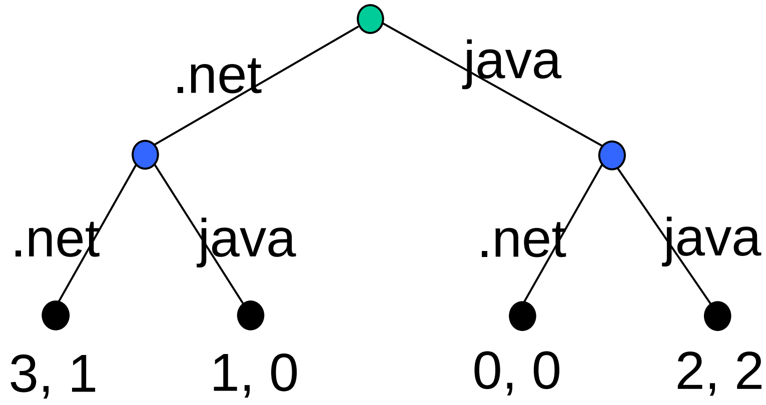
Mozilla

Microsoft		NN	NJ	JN	JJ
	.net	**3,1**	**3,1**	1,0	1,0
	java	0,0	2,2	0,0	**2,2**

NE

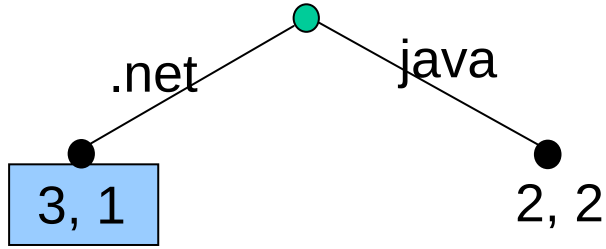
- Mozilla's JJ is a threat to Microsoft
- If loss of image is negligible, the threat is incredible
- Even if the threat is incredible, (java,JJ) is still a NE
  - How to get rid of this unconvincing NE?

Removing Incredible Threats and other poor NE



- Apply backward induction to game tree

- Single NE remains
  .net for Microsoft,
  .net, java for Mozilla



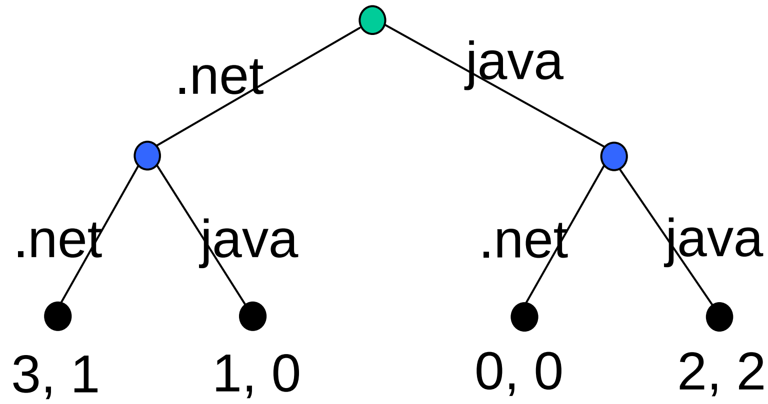
- In general, multiple NEs are possible after backward induction
  - cases with no strict preference over payoffs

- Corollary: be careful with reduction to normal form, when the game is not zero-sum!

Subgame Perfect Nash Equilibrium

- Def: a subgame is any subtree of the original game that also defines a proper game
  - only it makes sense in games with perfect information
- Def: a NE is *subgame perfect* if its restriction to *every* subgame is also a NE of the subgame
- The one deviation property: s^* is a Subgame Perfect Nash Equilibrium (SPNE) if and only if no player can gain by deviating from s^* in a single stage.
- Kuhn's Thr: every finite extensive form game with complete information has one SPNE
  - based on backward induction

NE and Incredible Threats



Mozilla

Microsoft	NN	NJ	JN	JJ
.net	3,1	3,1	1,0	1,0
java	0,0	2,2	0,0	2,2

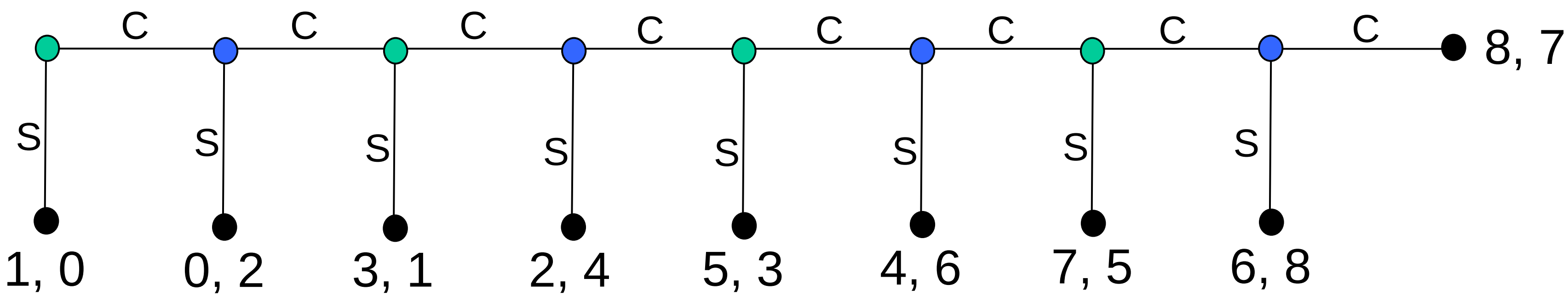
NE

SPNE

- JJ is an incredible threat and java-JJ is not an SPNE
- NN is not really a threat (it motivates more Microsoft to play net), but net-NN is not an SPNE

Weakness of SPNE
(or when GT does not predict people's behaviour)

- Centipede Game
  - two players alternate decision to continue or stop for k rounds
  - stopping gives better payoff than next player stopping in next round (but not if next player continues)



- Backward induction leads to unique SPNE
  - both players choose S in every turn
- How would you play this game with a stranger?
  - empirical evidence suggests people continue for many rounds

Stackelberg Game

- A particular game tree
- Two moves, leader then follower(s)
  - can be modeled by a game tree

- Stackelberg equilibrium
  - Leader chooses strategy knowing that follower(s) will apply best response
  - It is a SPNE for this particular game tree

Stackelberg Game and Computer Networking

- "Achieving Network Optima Using Stackelberg Routing Strategies."
  Yannis A. Korilis, Aurel A. Lazar, Ariel Orda. IEEE/ACM Transactions on Networking, 1997.

- "Stackelberg scheduling strategies".
  Tim Roughgarden. STOC 2001.

Promises

- Example: in a sequential prisoner's dilemma "I will not confess, if you not confess".

		Suspect 2	
		NC	C
Suspect 1	NC	2, 2	10, 1
	C	1, 10	5, 5

- Similar issues about credibility as for threats

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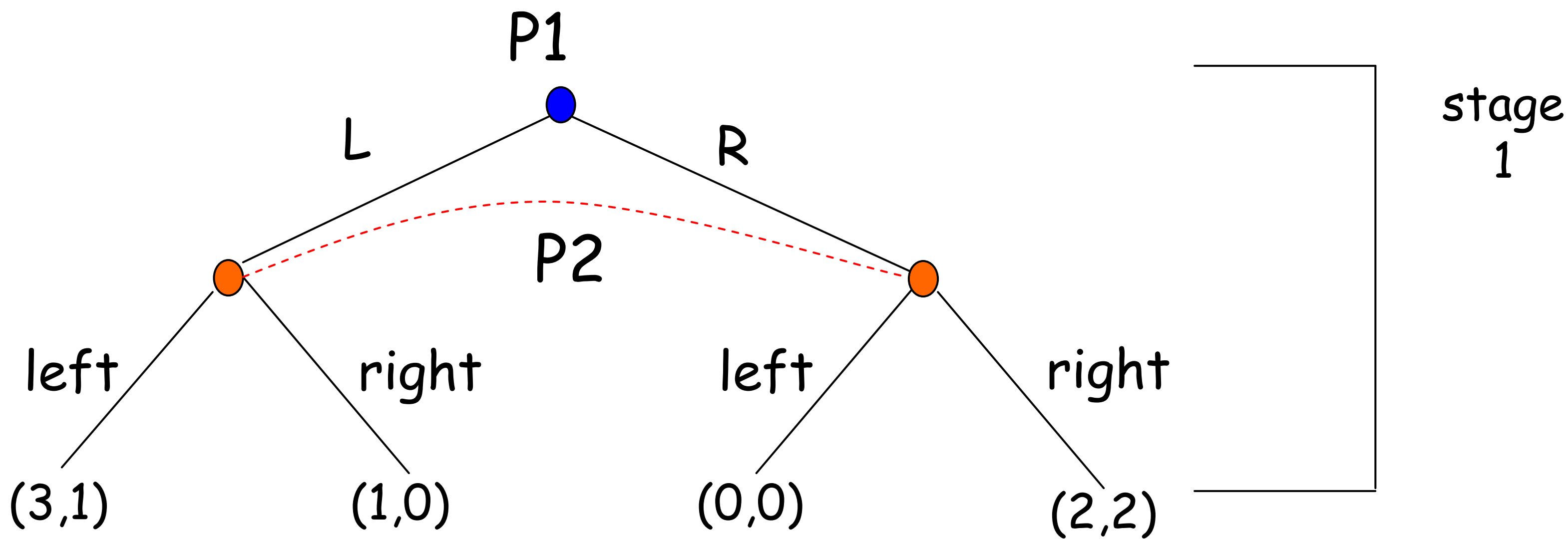
Repeated games

- players face the same "stage game" in every period, and the player's payoff is a weighted average of the payoffs in each stage.
- moves are simultaneous in each stage game.
- finitely repeated (finite-horizon) and infinitely repeated (infinite-horizon) games
- in this talk, we assume:
  - players perfectly observed the actions that had been played.

Repeated games are game trees

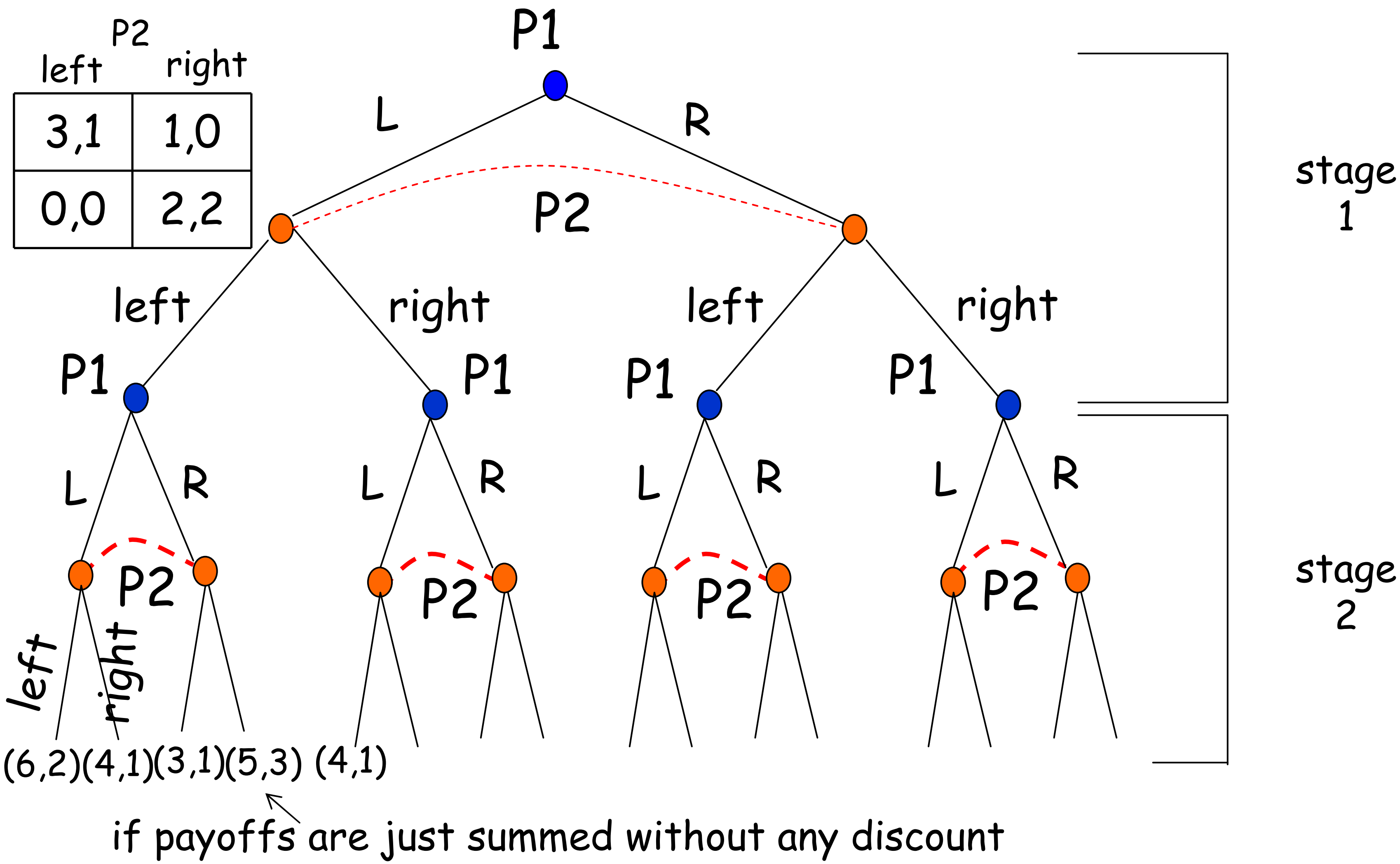
		P2 left	P2 right
P1	L	3,1	1,0
	R	0,0	2,2

- normal form simultaneous game
  - transform it in a game tree



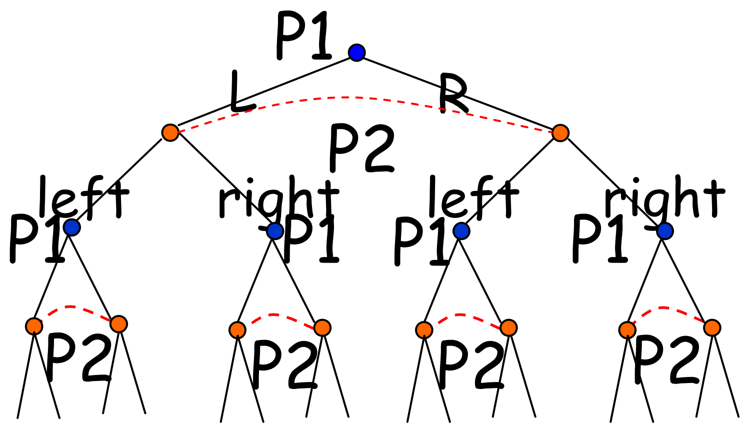
Repeated games are game trees

		P2 left	P2 right
P1	L	3,1	1,0
	R	0,0	2,2



Repeated games

- $A_i=(a_{i1}, a_{i2}, \ldots, a_{i|Ai|})$: action space for player i at each stage.
- $a^t=(a_1{}^t,\ldots, a_n{}^t)$: the actions that are played in stage t.
- $h^t=(a^0, a^1\ldots, a^{t-1})$: the history of stage t, the realized choices of actions at all stages before t.
- As common in game trees a pure strategy s_i for player i maps all its information sets to actions a_i in A_i
  - in this case it means mapping possible stage-t histories h^t to actions a_i in A_i
  - player strategy needs to specify his actions also after histories that are **impossible** if he carries out his plan (see Osborne and Rubinstein section 6.4)

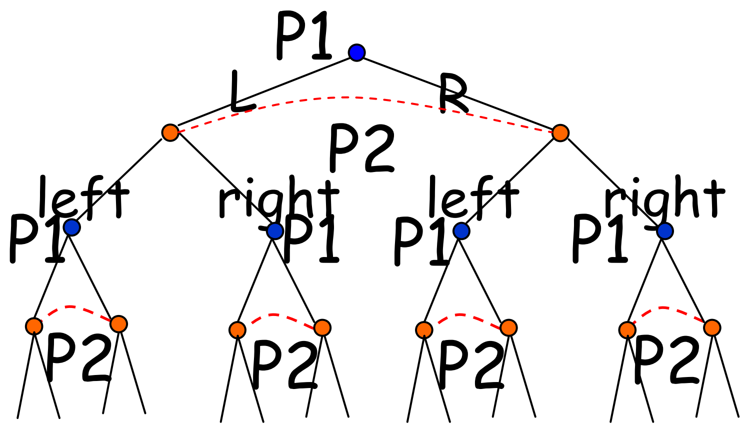


5 possible information sets and two actions available for each player.

- player 1 has 2^5 pure strategies
- player 2 has 2^5 pure strategies

Repeated games

- A mixed strategy x_i is a probability distribution over all possible pure strategies.
- A behavioral strategy b_i is a function which assigns to each information set a probability distribution over available actions, that is, randomizing over the actions available at each node.
  - see Osborne and Rubinstein, section 11.4

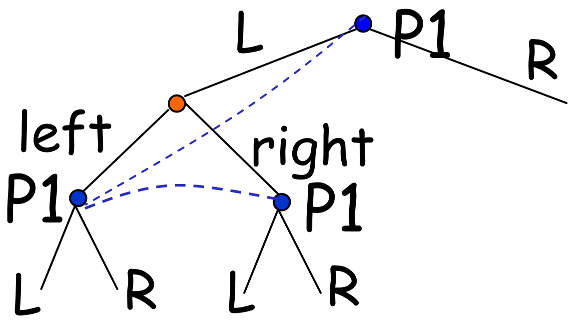


5 possible information sets and two actions available for each player.

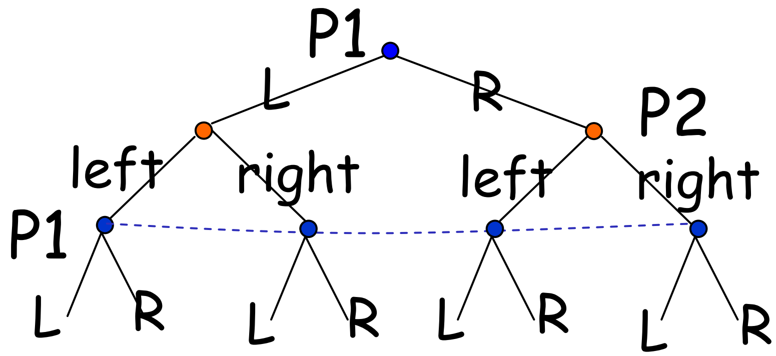
- a mixed strategy for player 1 is specified by 2^5-1 values in [0,1]
- a behavioral strategy for player 1 is specified by 5 values in [0,1]

Repeated games

- behavioral strategies are outcome-equivalent to mixed strategies and vice versa in games with perfect recall,
  - perfect recall=a player remembers whatever he knew in the past
- two games with imperfect recall
  1. P1 forgets that he has already played):
  2. P1 forgets what he played



P1 behavioral strategy: play L with prob. p
• can give LL with prob. p^2, LR with prob. p(1-p)
P1 pure strategies: play L and play R
• no mixed strategy can be outcome equivalent to the behavioral strategy



A possible P1 mixed strategy: play LL with prob. 1/2, RR with prob. 1/2
P1 behavioral strategy: 1st time play L with prob. p, 2nd time play L with prob. q
• can give LL with prob. pq,
RR with prob. (1-p)(1-q)
• not possible to obtain the mixed strategy

Infinite-horizon games

- stage games are played infinitely.
- payoff to each player is the sum of the payoffs over all periods, weighted by a discount factor δ, with $0 < \delta < 1$.
  - δ can be interpreted also as the probability to continue the game at each stage ($1-\delta$ is the prob. to stop playing)

- Central result: Folk Theorem.

Nash equilibrium in repeated game

- We may have new equilibrium outcomes that do not arise when the game is played only once.

  - Reason: players' actions are observed at the end of each period, players can condition their play on the past play of their opponents.

  - Example: cooperation can be a NE in Prisoner's Dilemma Game in infinitely repeated game.

Finite-horizon Prisoner's dilemma

Prisoner's Dilemma Game (Payoff Matrix)		P2	
		Cooperate	Defect
P1	Cooperate	5, 5	-3, 8
	Defect	8, -3	0, 0

- A Prisoner's Dilemma game is played 100 times.
- At the last play, $h=2^{99}x2^{99} \approx 4x10^{59}$ histories, so there are 2^h pure strategies !
- One unique subgame perfect NE: always "defect"
  - same criticism that for the centipede game (people play differently)

Infinite-horizon Prisoner's Dilemma

Prisoner's Dilemma Game (Payoff Matrix)		P2	
		Cooperate	Defect
P1	Cooperate	5, 5	-3, 8
	Defect	8, -3	0, 0

- How to find Nash equilibrium?
  - we cannot use Backward induction.

- Let's guess: trigger strategy can be subgame perfect NE if δ (discount factor) is close to one.

Trigger Strategy

- Def: follow one course of action until a certain condition is met and then follow a different strategy for the rest of the repeated game.
- Idea: each player will be deterred from abandoning the cooperative behavior by being punished. Punishments from other player are triggered by deviations
- examples:
  - trigger strategy 1: I cooperate as long as the other player cooperates, and I defect forever if the other player defects in one stage.
  - trigger strategy 2: I alternates C, D, C, … as long as the other player alternates D, C, D, … , if the other player deviates from this pattern, then I deviate forever.

Infinite-horizon Prisoner's Dilemma

- Trigger strategy 1: cooperate as long as the other player cooperates, and defect forever if the other player defects in one stage.
- Trigger strategy 1 can be subgame perfect NE if the discount factor δ is close to one.

  Proof:
  - if both players cooperate, then payoff is $5/(1-\delta)=5*(1+\delta+\delta^2+\ldots)$
  - suppose one player could defect at some round, in order to discourage this behavior, we need $5/(1-\delta) \geq 8$, or $\delta \geq 3/8$.
  - so, as long as $\delta \geq 3/8$, the pair of trigger strategies is subgame perfect NE

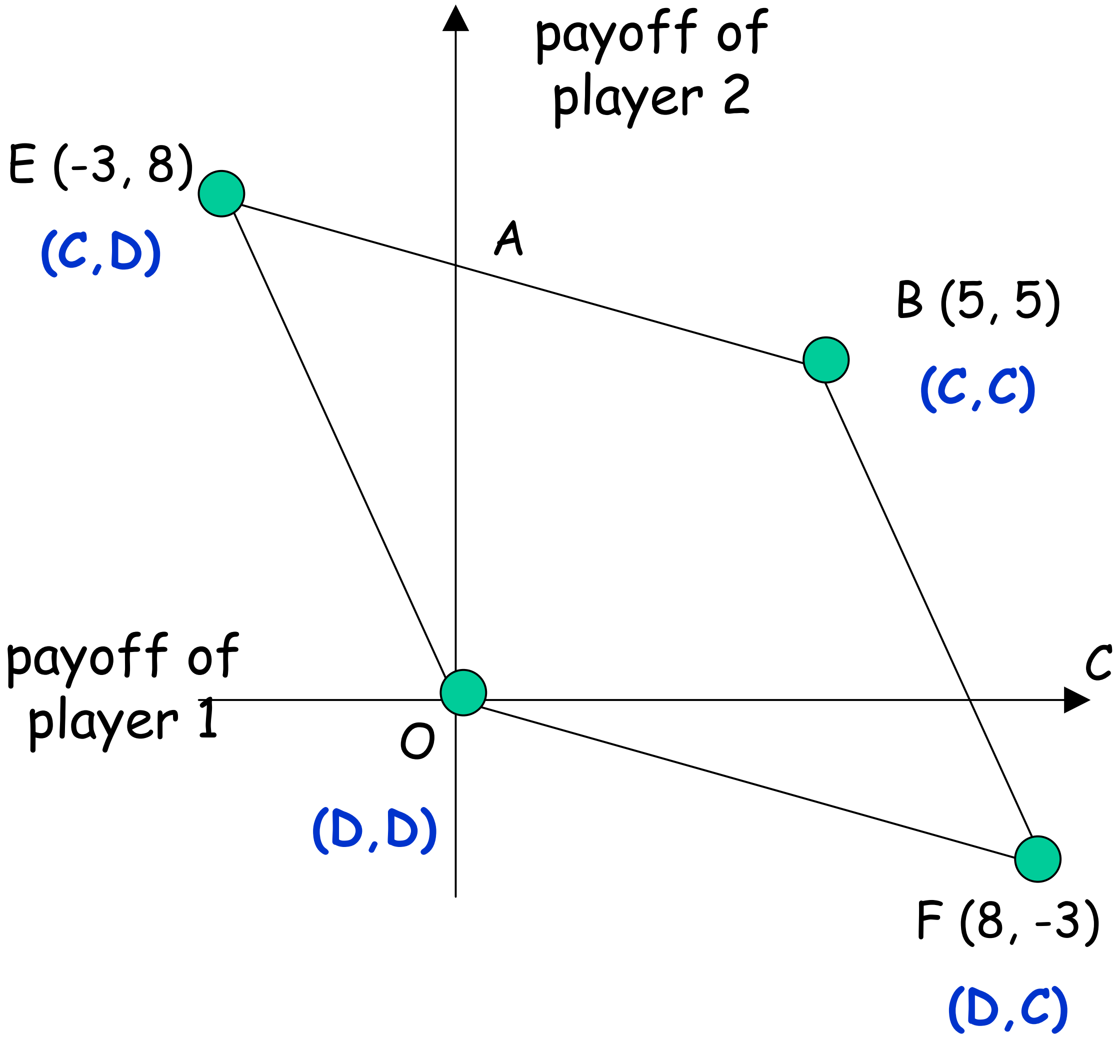
Cooperation can happen at Nash equilibrium !

Infinite-horizon Prisoner's Dilemma

- Trigger strategy 2: player 1 alternates C, D, C, … as long as player 2 alternates D, C, D, … , if player 2 deviates from this pattern, then player 1 deviates forever. This is also true for player 2.
- This pair of trigger strategies is also subgame perfect NE if δ is sufficiently close to one.

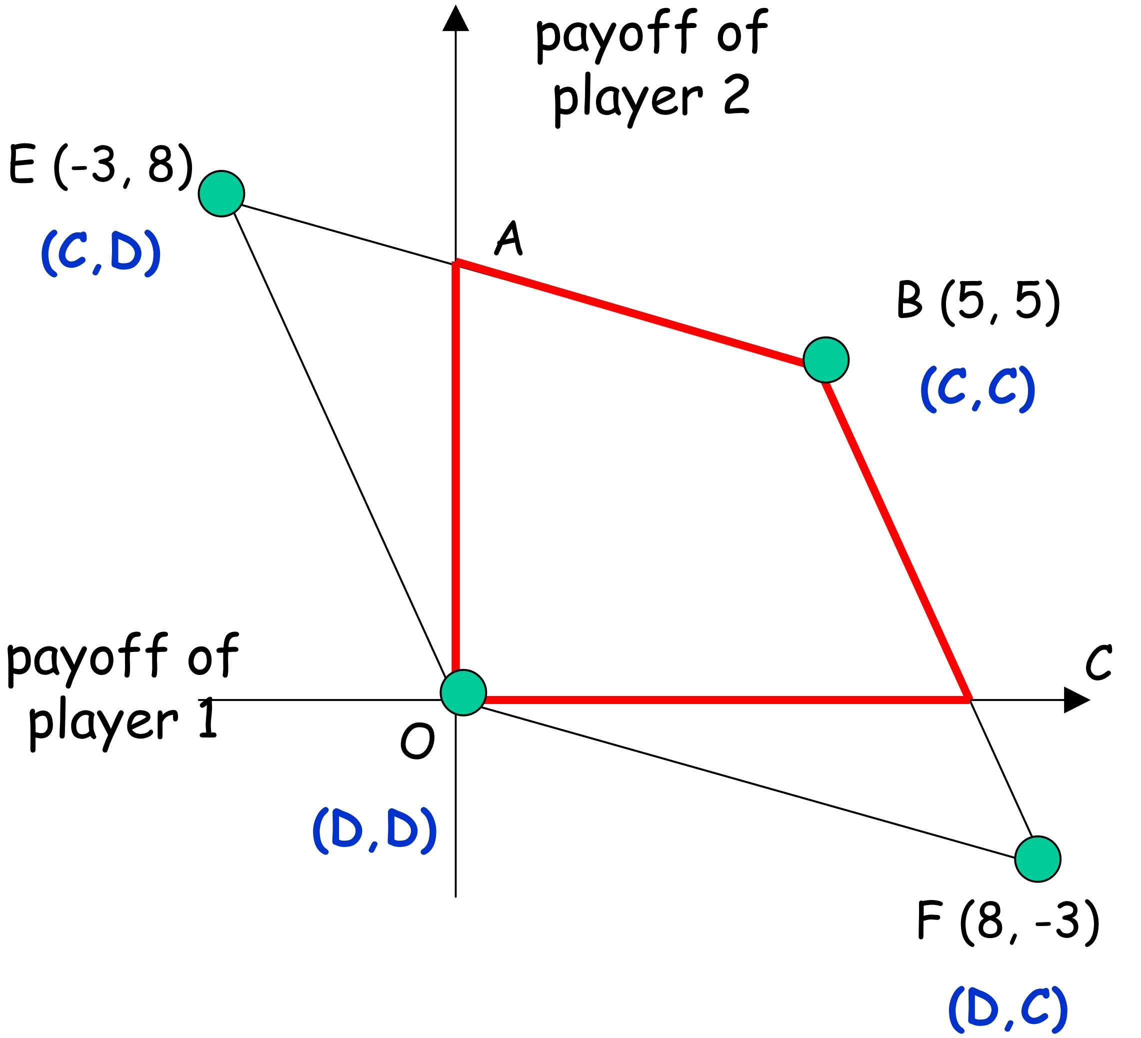
- In fact, there are lots of subgame perfect NEPs if δ is sufficiently close to one.
- What is happening here?

Infinite-horizon Prisoner's Dilemma



Region EOFBE contains the payoffs of all possible mixed strategy pairs.

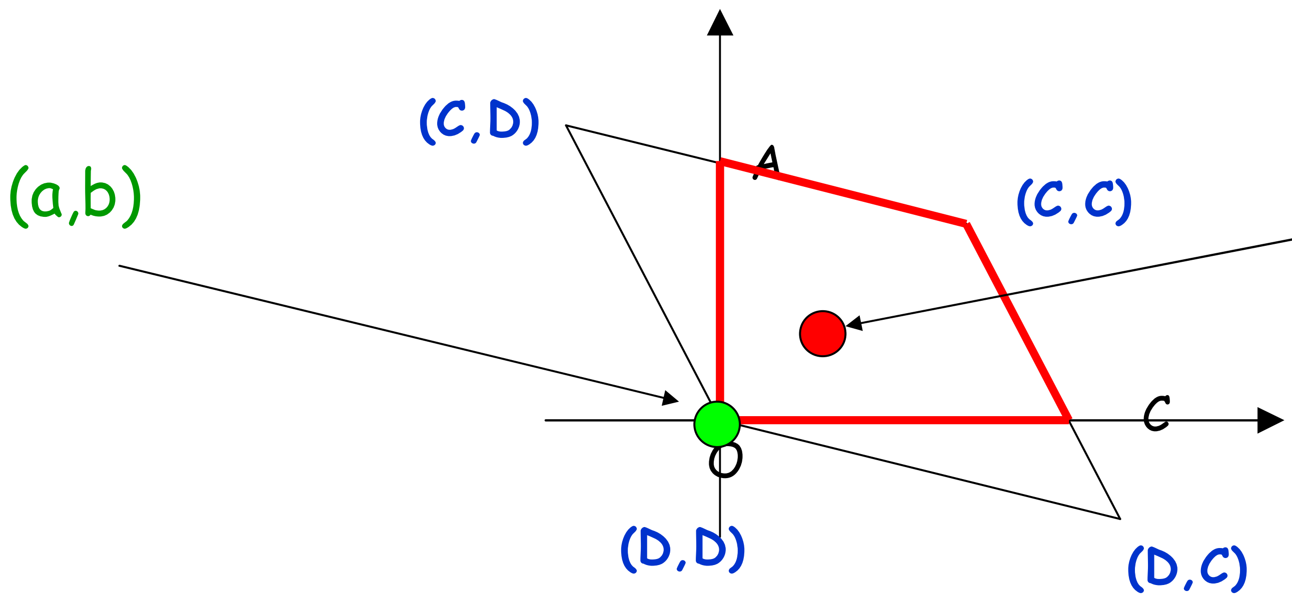
Infinite-horizon Prisoner's Dilemma



Any point in the region OABC can be sustained as a subgame perfect NE of the repeated game given the discount factor of the players is close to one (that is, players are patient enough) !

Folk Theorem

- For any two-player stage game with a Nash equilibrium with payoffs (a, b) to the players. Suppose there is a pair of strategies that give the players (c, d). Then, if c>=a and d>=b, and the discount factors of the players are sufficiently close to one, there is a subgame perfect NE with payoffs (c, d) in each period.



Axelrod's tournament (1984)

- competition of software playing Repeated Prisoner's dilemma
- 1st competition: the winner is a 4 line program "Tit For Tat" (TFT)
  - cooperate at the first stage
  - copy what your opponent has done at the previous stage
- 2nd competition: again Tit For Tat
- Good properties of TFT
  - nice, starts cooperating
  - retaliatory, punishes defection
  - forgiving
  - clear

Applications of repeated games in computer networking

- "Optimal Routing Control: Repeated Game Approach",
  R. La and V. Anantharam. IEEE Tans. on Automatic Control, 2002.

- "Cooperation in Wireless Ad Hoc Networks."
  V. Srinivasan, P. Nuggehalli, C. Chiasserini, and R. Rao. IEEE Infocom 2003.

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Evolutionary games

- Maynard Smith and Price in 1973
- game theoretical look at species evolution
- new equilibrium concept (static analysis):
  - Evolutionary Stable Strategies (ESS)
    - a refinement of NE
- also dynamics (e.g. replicator dynamics)
- In this talk we only consider symmetric games

	P2	
P1	a,a	b,c
	c,b	d,d

Hawk-Dove game

- rules
  - resource to be assigned with value 50
  - hawk attacks until it lets opponent escape or it is injured (-100 damage)
  - dove at most engages in symbolic conflict (at most -20 damage for time needed)

	H	D
H	-25,-25	50,0
D	0,50	15,15

Hawk-Dove game

- target
  - how would these two species evolve? (or these two behaviors among members of the same species?)
  - how would be the equilibrium? only a single species/behavior surviving? a mix of them?
- assumptions
  - species with higher payoff will have higher chance to reproduce and their percentage will increase
  - random matching between individuals

	H	D
H	-25,-25	50,0
D	0,50	15,15

Hawk-Dove game

- Can a population of hawks be *invaded* by a *mutant* acting as a dove?
  - Yes, "everyone hawk" is not an ESS
- Can a population of doves be invaded by a mutant acting as a hawk?
  - Yes, "everyone dove" is not an ESS

	H	D
H	-25,-25	50,0
D	0,50	15,15

Hawk-Dove game

- Is there a stable mix of hawk and doves?
  - what about ½ hawks and ½ doves?
    - not stable, still convenient to be hawk
  - 7/12 hawks, 5/12 doves
    - This is an ESS! But also a NE
- Interpretation, it is stable a population
  - with 7/12 pure hawks and 5/12 pure doves
  - homogeneous individuals acting 7/12 of time as hawk and 5/12 as dove
  - heterogeneous individuals, but with this average behavior

	H	D
H	-25,-25	50,0
D	0,50	15,15

A more formal definition

- an incumbent strategy $x \in \Delta$, where Δ is the set of all mixed strategies,
  - x_i - population share of pure strategy i.
  - e^i - pure strategy i.
  - $u(e^i, e^j)$ - matrix payoffs
  - $u(e^i, x)$ - expected payoff (fitness) of strategy i at a random match when the population is in state $x=(x_1, \ldots, x_n)$.

$$u(e^i, x) = \sum_j u(e^i, e^j) x_j$$

- a single mutant playing strategy $y \in \Delta$, would get the payoff:

$$u(y, x) = \sum_i y_i u(e^i, x) = \sum_{i,j} y_i u(e^i, e^j) x_j$$

A more formal definition

- an incumbent strategy $x \in \Delta$, where Δ is the set of all mixed strategies,
- a mutant strategy $y \in \Delta$. The share of mutants in the population is ε, where $\varepsilon \in (0,1)$.
- random matching of players, then the payoff of a generic player is the same as in a match with a player who plays mixed strategy $w = \varepsilon y+(1- \varepsilon)x \in \Delta$.
- A strategy x is an evolutionarily stable strategy (ESS) if the following inequality is true for any mutant strategy $y \neq x$ for a given value of ε,

$$\boxed{u(x, \epsilon y + (1-\epsilon)x) > u(y, \epsilon y + (1-\epsilon)x)}$$

Characterization of ESS

Equivalently, x is an ESS if and only if it meets the following conditions

- first-order best-response condition:

  x is the best response to itself.

$$\boxed{u(y,x) \leq u(x,x) \quad \forall y \in \triangle}$$

- second-order best-response condition:

  x is a better response to a mutant strategy y (y≠x), if y has the same payoff as x when playing against x.

$$\boxed{u(y,x) = u(x,x) \Rightarrow u(y,y) < u(x,y)}$$

Homework: Prove it!

Some immediate results…

- $\Delta^{ESS} \subset \Delta^{NE}$
  - follows from the first-order condition
  - Δ^{ESS} – the set of evolutionarily stable strategies
  - Δ^{NE} – the set of NE strategies

- if (x, x) is a strict Nash equilibrium, then x is evolutionarily stable.
  - x is a strict Nash equilibrium if x is the only best response to itself.
- ESS does not in general imply that the average population fitness is maximized.

  ESS does not imply social efficiency.

Example 1

Prisoner's Dilemma (Payoff Matrix)		P2	
		Cooperate	Defect
P1	Cooperate	5, 5	-3, 8
	Defect	8, -3	0, 0

- Prisoner's Dilemma Game: Nash equilibrium strategy Defect is the unique best reply to any strategy y, so Defect-Defect is a strict Nash equilibrium, then it is ESS.

Example 2

Coordination Game (Payoff Matrix)		P2	
		right	left
P1	right	2, 2	0, 0
	left	0, 0	1, 1

Two pure strategy NEs (right, right), (left, left) and a mixed strategy NE (determine it)

- Coordination Game: both pure strategy Nash equilibria are ESS.
  - in fact, both NEs are strict Nes.
  - Socially efficient NE strategy right is a ESS.
  - Socially inefficient NE strategy left is also a ESS.

Example 3

- Rock-Scissors-Paper (RSP) Game:

RSP Game (Payoff Matrix)		P2		
		Rock	Scissors	Paper
P1	Rock	0, 0	1, -1	-1, 1
	Scissors	-1, 1	0, 0	1, -1
	Paper	1, -1	-1, 1	0, 0

- Unique NE strategy x=(1/3, 1/3, 1/3) is NOT ESS !
- check condition 1: all strategies are best replies to x.
- check condition 2: mutant y=(1, 0, 0), u(y, y)=1=u(x, y), NOT satisfied

Dynamics

- ESS is an equilibrium, there can be multiple ones
- Starting from a generic population composition, do we converge to a ESS? to which one?
- A possible dynamics is replicator dynamics

Replicator Dynamics: Model

$$\frac{dx_i}{dt} = (u(e^i, x) - u(x, x))x_i$$

x_i – population share of pure strategy i.

e^i – pure strategy i.

$u(e^i, x)$ – expected payoff (fitness) of strategy i at a random match when the population is in state $x=(x_1, \ldots, x_n)$.

$u(x, x)$ – population expected payoff (fitness) is the expected payoff to an individual drawn at random from the population:

$$u(x,x) = \sum_{i=1}^{k} x_i u(e^i, x)$$

Stability Concepts

- Lyapunov Stability: a state x is "stable" or "Lyapunov stable" if no small perturbation of the state induces a movement away from x.
  - no push away from x
- Asymptotical Stability: a state x is "asymptotical stable" if it is "Lyapunov stable" and all sufficiently small perturbations of the state induce a movement back toward x.
  - an ESS is asymptotical stable

Normalization of 2x2 Symmetric Game

General 2 x 2 Symmetric (Payoff Matrix)		P2	
		L	R
P1	L	a11	a12
	R	a21	a22

$a1 = a11 - a21$, $a2 = a22 - a12$

a1	0
0	a2

This local shift in payoff produces a new game which has the same set of NEPs as the original game.

✓ does not change a player's pure and mixed best-reply correspondences.

✓ does not change weak and strict dominance orderings.

NE, ESS in Symmetric 2x2 Game

	L	R
L	a1	0
R	0	a2

mixed strategy: $x=\lambda L+(1-\lambda)R$

with $\lambda = a2/(a1+a2)$

a2

I — Prisoner's Dilemma Game

NE: (R,R)

ESS: (R,R)

II — Coordination Game

NE: (R,R), (L,L), (x,x)

ESS: (R,R), (L,L)

III — Hawk-dove Game

NE: (L,R), (R,L), (x,x)

ESS: (x,x)

IV — Prisoner's Dilemma Game

NE: (L,L)

ESS: (L,L)

a1

Replicator Dynamics

$$\dot{x}_1 = (a_1 x_1 - a_2 x_2) x_1 x_2$$

NE, ESS, Replicator Dynamics in Symmetric 2x2 Game

x_1 – probability of playing "L"; x – mixed strategy defined before

a2

Prisoner's Dilemma Game

NE: (R,R)

ESS: (R,R)

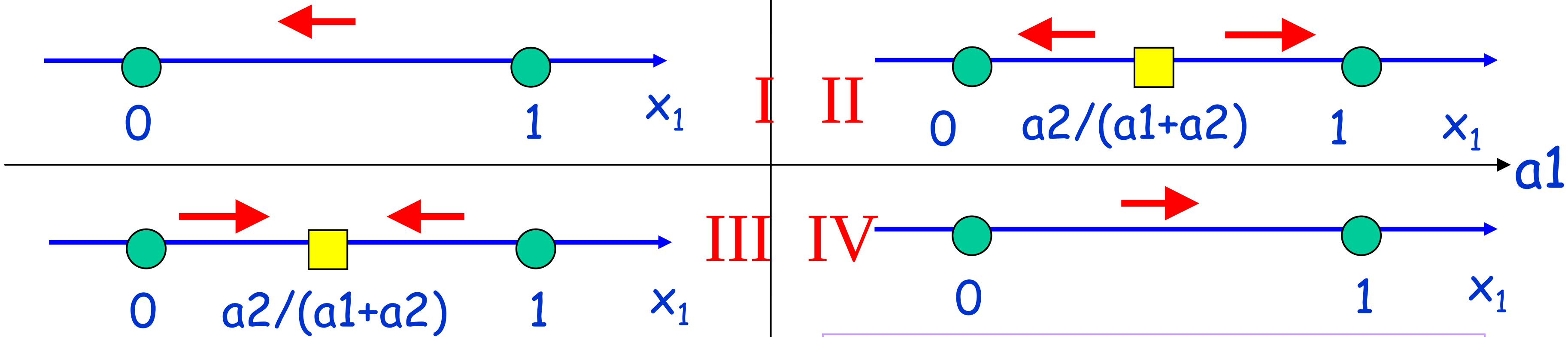
Replicator Dynamics → ESS=NE

Coordination Game

NE: (R,R), (L,L), (x,x)

ESS: (R,R), (L,L)

Replicator Dynamics → ESS



Hawk-dove Game

NE: (L,R), (R,L), (x,x)

ESS: (x,x)

Replicator Dynamics → ESS

Prisoner's Dilemma Game

NE: (L,L)

ESS: (L,L)

Replicator Dynamics → ESS=NE

Rock-Scissors-Paper (RSP) Game

- Unique NE strategy x=(1/3, 1/3, 1/3) is NOT ESS !

RSP Game (Payoff Matrix)		P2		
		Rock	Scissors	Paper
P1	Rock	0, 0	1, -1	-1, 1
	Scissors	-1, 1	0, 0	1, -1
	Paper	1, -1	-1, 1	0, 0

- How about the Replicator Dynamics?

Example: Rock-Scissors-Paper (RSP) Game

RSP Game		P2		
		R	S	P
P1	R	0, 0	1, -1	-1, 1
	S	-1, 1	0, 0	1, -1
	P	1, -1	-1, 1	0, 0

A is the payoff matrix of one player

0	1	-1
-1	0	1
1	-1	0

Replicator Dynamics:

$$\frac{dx_i}{dt} = (u(e^i, x) - u(x, x))x_i$$

$$\boxed{\dot{x}_1 = (+x_2 - x_3 - x'Ax)x_1}$$

$$\dot{x}_2 = (-x_1 + x_3 - x'Ax)x_2$$

$$\dot{x}_3 = (+x_1 - x_2 - x'Ax)x_3$$

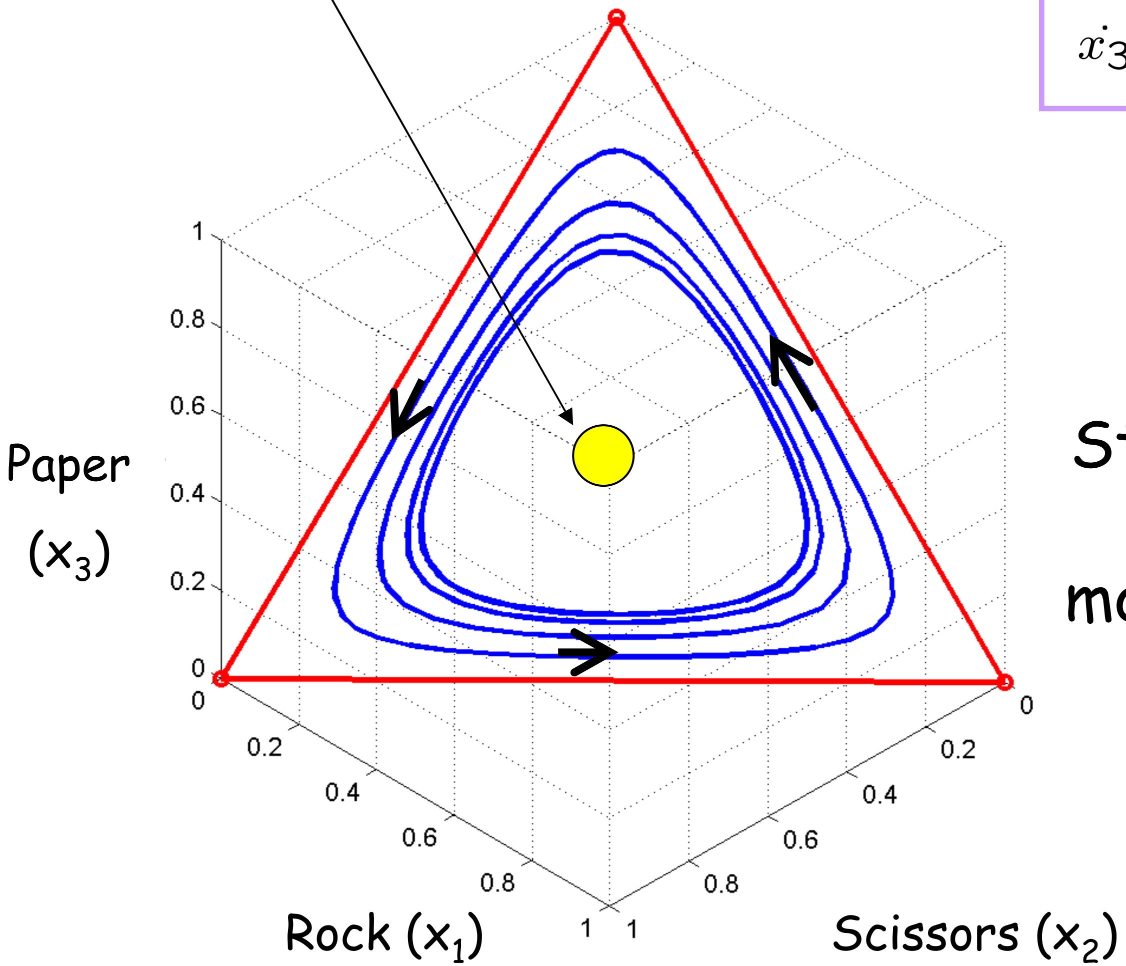
Rock-Scissors-Paper (RSP) Game

NE strategy x=(1/3, 1/3, 1/3), but not ESS

Replicator Dynamics:

$$\dot{x}_1 = (+x_2 - x_3 - x'Ax)x_1$$
$$\dot{x}_2 = (-x_1 + x_3 - x'Ax)x_2$$
$$\dot{x}_3 = (+x_1 - x_2 - x'Ax)x_3$$

NE strategy is Lyapunov stable, but not asymptotically stable



Start from any initial state, the system moves forever along a closed curve!

Evolutionary Game Theory and Computer Networking

- "An evolutionary game perspective to ALOHA with power control".
  E. Altman, N. Bonneau, M. Debbah, and G. Caire. Proceedings of the 19th International Teletraffic Congress, 2005.

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  D.S. Menasche, D.R. Figueiredo, E. de Souza e Silva. Performance Evaluation 62 (2005)