Game Theory: introduction and applications to computer networks

Lecture 4: N-person non zero-sum games

Giovanni Neglia
INRIA – EPI Maestro

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Slides are based on a previous course
with D. Figueiredo (UFRJ) and H. Zhang (Suffolk University)

Outline

- ❒ Two-person zero-sum games
- ❒ Two-person non-zero-sum games
- ❒ N-persons games
  - ❍ Overview (easy or difficult games)
  - ❍ Cooperative games
    - games in characteristic function form
    - which coalitions should form?

2-Person Games

- Zero-Sum Games (ZSG)
  - nice equilibria: unique value of the game, interchangeable strategies,... (see minimax theorem)

Player 2

		A	B	C	D	
Player 1	A	3	2	2	5	2
	B	2	-10	0	-1	-10
	C	5	2	2	3	2
	D	8	0	-4	-5	-5
		8	2	2	5	

- Non-Zero-Sum Games
  - Nash equilibrium is sometimes unattractive: multiple non-equivalent NE, not Pareto optimal,...

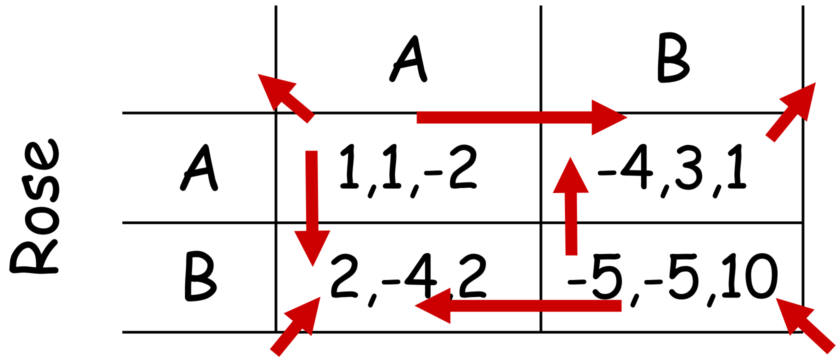
N-Person Games

- Same distinction?
- No, N-Person Zero-Sum Games are difficult too!

A 2x2x2 game

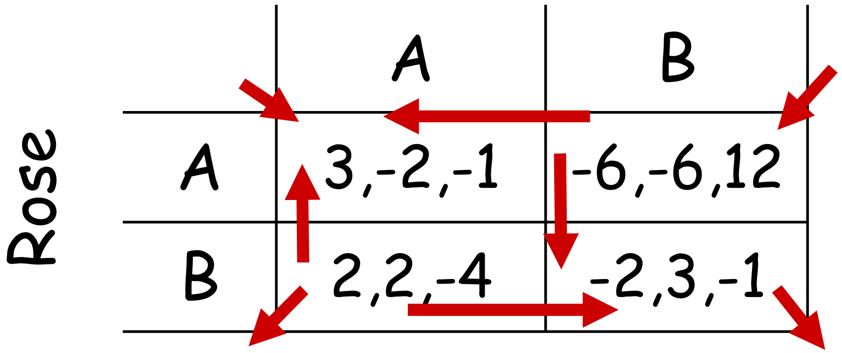
(let's meet Rose, Colin and Larry)

Larry A



Rose \ Colin	A	B
A	1,1,-2	-4,3,1
B	2,-4,2	-5,-5,10

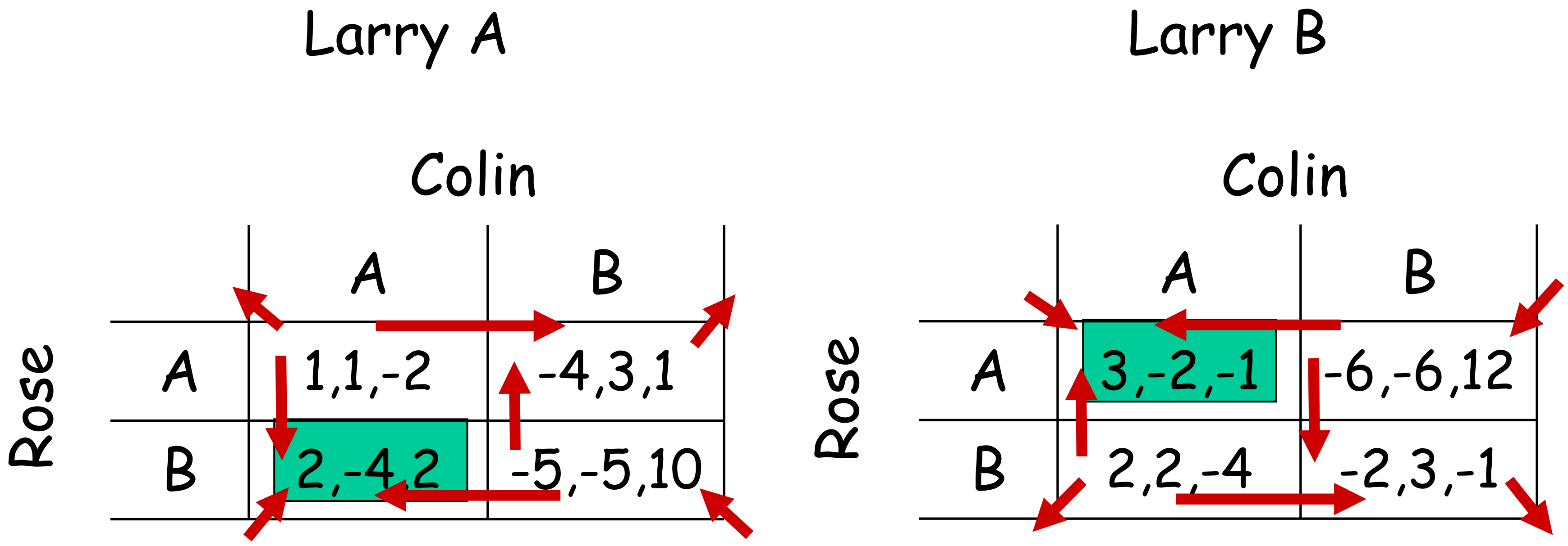
Larry B



Rose \ Colin	A	B
A	3,-2,-1	-6,-6,12
B	2,2,-4	-2,3,-1

A 2x2x2 game

(let's meet Rose, Colin and Larry)



Larry A

Colin

Rose

	A	B
A	1,1,-2	-4,3,1
B	2,-4,2	-5,-5,10

Larry B

Colin

Rose

	A	B
A	3,-2,-1	-6,-6,12
B	2,2,-4	-2,3,-1

- Two pure strategy equilibria: (B,A,A) (A,A,B)
  - not equivalent
  - not interchangeable

A new possibility: Coalitions

- Larry and Colin against Rose
  - a 2-Person Zero-Sum game

Larry A

Colin

		A	B
Rose	A	1,1,-2	-4,3,1
	B	2,-4,2	-5,-5,10

Larry B

Colin

		A	B
Rose	A	3,-2,-1	-6,-6,12
	B	2,2,-4	-2,3,-1

Colin & Larry

		AA	AB	BA	BB
Rose	A				
	B				

A new possibility: Coalitions

- Larry and Colin against Rose
  - a 2-Person Zero-Sum game

Larry A

Colin

		A	B
Rose	A	1,1,-2	-4,3,1
	B	2,-4,2	-5,-5,10

Larry B

Colin

		A	B
Rose	A	3,-2,-1	-6,-6,12
	B	2,2,-4	-2,3,-1

Colin & Larry

		AA	BA	AB	BB
Rose	A	1			
	B				

A new possibility: Coalitions

- Larry and Colin against Rose
  - a 2-Person Zero-Sum game

Larry A

Colin

		A	B
Rose	A	1,1,-2	-4,3,1
	B	2,-4,2	-5,-5,10

Larry B

Colin

		A	B
Rose	A	3,-2,-1	-6,-6,12
	B	2,2,-4	-2,3,-1

Colin & Larry

		AA	BA	AB	BB
Rose	A	1		3	
	B				

A new possibility: Coalitions

- Larry and Colin against Rose
  - a 2-Person Zero-Sum game

Larry A

Colin

		A	B
Rose	A	1,1,-2	-4,3,1
	B	2,-4,2	-5,-5,10

Larry B

Colin

		A	B
Rose	A	3,-2,-1	-6,-6,12
	B	2,2,-4	-2,3,-1

Colin & Larry

		AA	BA	AB	BB
Rose	A	1	-4	3	-6
	B	2	-5	2	-2

A new possibility: Coalitions

- Larry and Colin against Rose
  - optimal (mixed) strategies

Colin & Larry

		AA	BA	AB	BB	
Rose	A	1	-4	3	-6	**3/5**
	B	2	-5	2	-2	**2/5**
			4/5		**1/5**	

  - Rose's payoff *(security level)*: -4.40
  - What about Colin & Larry?
  - Considering again the original game:
    Colin -0.64, Larry 5.04

A new possibility: Coalitions

- Larry and Colin against Rose
  - R=-4.40, C=-0.64, L=5.04
- Rose and Larry against Colin
  - R=2.00, C=-4.00, L=2.00
- Rose and Colin against Larry
  - R=2.12, C=-0.69, L=-1.43
- Which coalition will form?
  - Rose wants Colin
  - Larry wants Colin
  - Colin wants Larry
  - Answer: Colin & Larry against Rose
- Nothing else?
  - It can happen that no pair of players prefer each other!

Sidepayments

- Current winning coalition:
  - Larry and Colin against Rose
    - R=-4.40, C=-0.64, L=5.04
- Rose's best coalition:
  - Rose and Colin against Larry
    - R=2.12, C=-0.69, L=-1.43
- What if Rose offers 0.1 to Colin to form a coalition?
  - R=2.02, C=-0.59, L=-1.43
  - It would be better also for Colin

Theory of cooperative games with sidepayments

- It starts with von Neumann and Morgenstern (1944)
- Two main (related) questions:
  - which coalitions should form?
  - how should a coalition which forms divide its winnings among its members?
- The specific strategy the coalition will follow is not of particular concern...
- Note: there are also cooperative games without sidepayments

Theory of cooperative games with sidepayments

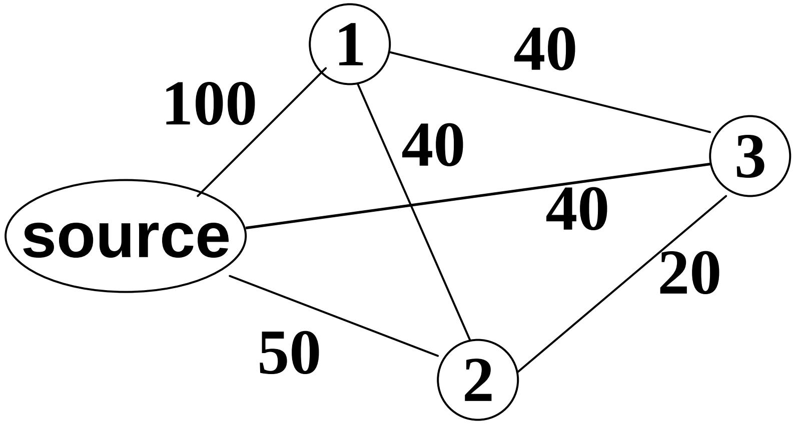
- **Def**. A game in *characteristic function form* is a set N of players together with a function v() which for any subset S of N (a coalition) gives a number v(S) (the value of the coalition)
- The interesting characteristic functions are the superadditive ones, i.e.
  - v(S U T) ≥ v(S) + v(T), if S and T are disjoint

Example 1: our game

- Larry&Colin vs Rose: R=-4.40, C=-0.64, L=5.04
- Rose&Larry vs Colin: R=2.00, C=-4.00, L=2.00
- Rose&Colin vs Larry: R=2.12, C=-0.69, L=-1.43
- The characteristic function
  - v(void)=0
  - v({R})=-4.40, v({C})=-4.00, v({L})=-1.43,
  - v({R,C})=1.43, v({R,L})=4.00, v({C,L})=4.40,
  - v({R,L,C})=0
- **Remark 1**: Any Zero-Sum game in normal form can be translated into a game in characteristic form
- **Remark 2**: Also Non-Zero-Sum games can be put in this form, but it could be **not** an accurate reflection of the original game

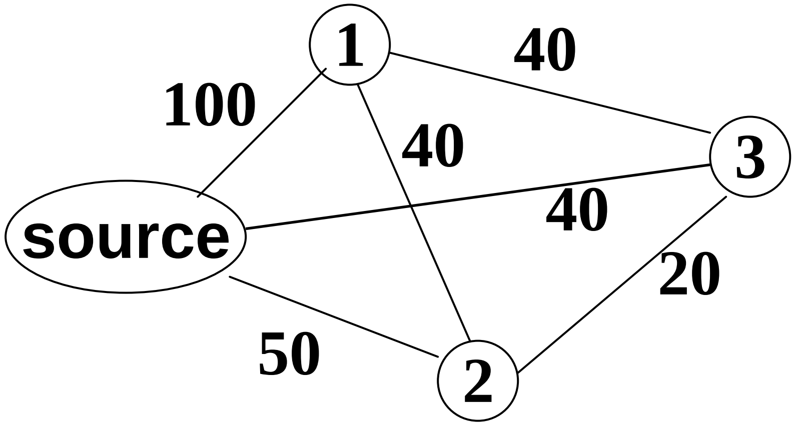
Example 2:
Minimum Spanning Tree game

- For some games the characteristic form representation is immediate
- Communities 1,2 & 3 want to be connected to a nearby power source
  - Possible transmission links & costs as in figure



Example 2:
Minimum Spanning Tree game

- Communities 1,2 & 3 want to be connected to a nearby power source

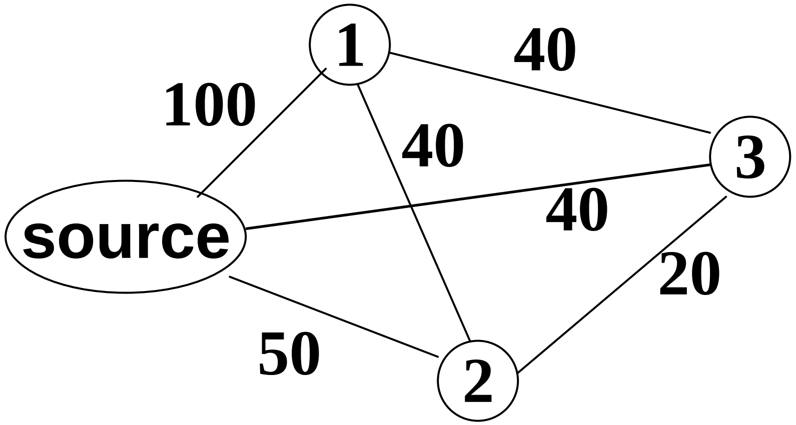


v(void) = 0
v(1) = -100
v(2) = -50
v(3) = -40
v(12) = -90
v(13) = -80
v(23) = -60
v(123) = -100

A normalization can be done

Example 2:
Minimum Spanning Tree game

- Communities 1,2 & 3 want to be connected to a nearby power source

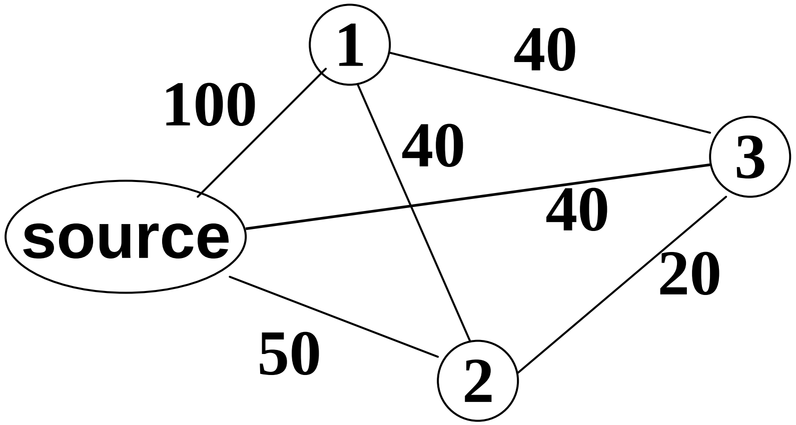


v(void) = 0
v(1) = -100 +100 = 0
v(2) = -50 + 50 = 0
v(3) = -40 + 40 = 0
v(12) = -90 + 100 + 50 = 60
v(13) = -80 + 100 + 40 = 60
v(23) = -60 + 50+ 40 = 30
v(123) = -100 + 100 + 50+ 40 = 90

A strategically equivalent game

Example 2:
Minimum Spanning Tree game

- Communities 1,2 & 3 want to be connected to a nearby power source



v(void) = 0/90 = 0
v(1) = (-100 +100)/90 = 0
v(2) = (-50 + 50)/90 = 0
v(3) = (-40 + 40)/90 = 0
v(12) = (-90 + 100 + 50)/90 = 2/3
v(13) = (-80 + 100 + 40)/90 = 2/3
v(23) = (-60 + 50+ 40)/90 = 1/3
v(123) = (-100 + 100 + 50+ 40)/90 = 1

A strategically equivalent game

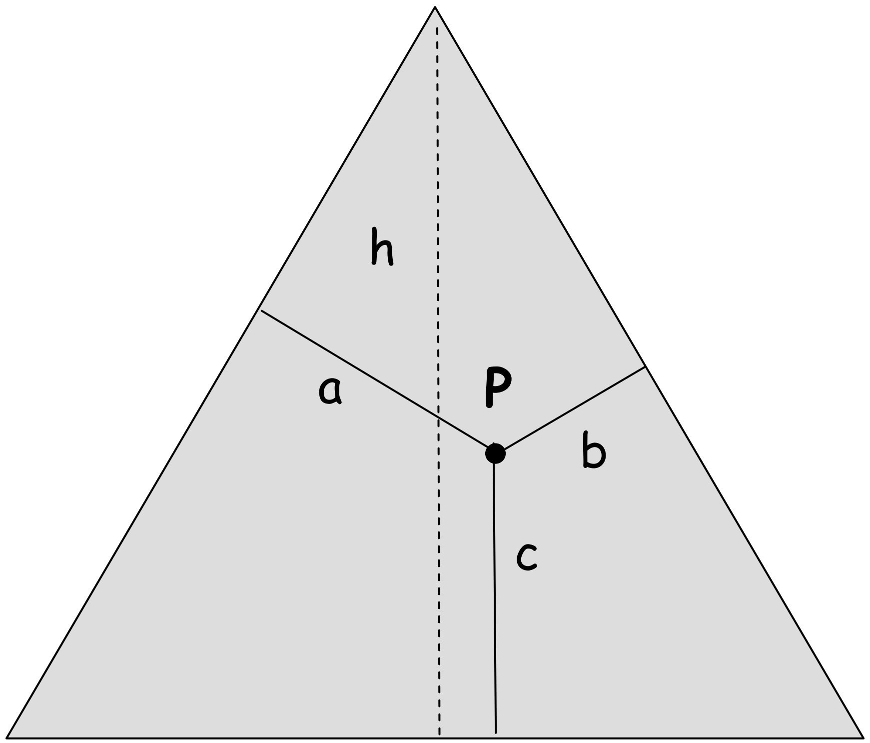
The important questions

- Which coalitions should form?
- How should a coalition which forms divide its winnings among its members?
- Unfortunately there is no definitive answer
- Many concepts have been developed since 1944:
  - stable sets
  - **core**
  - **Shapley value**
  - bargaining sets
  - nucleolus
  - Gately point

Imputation

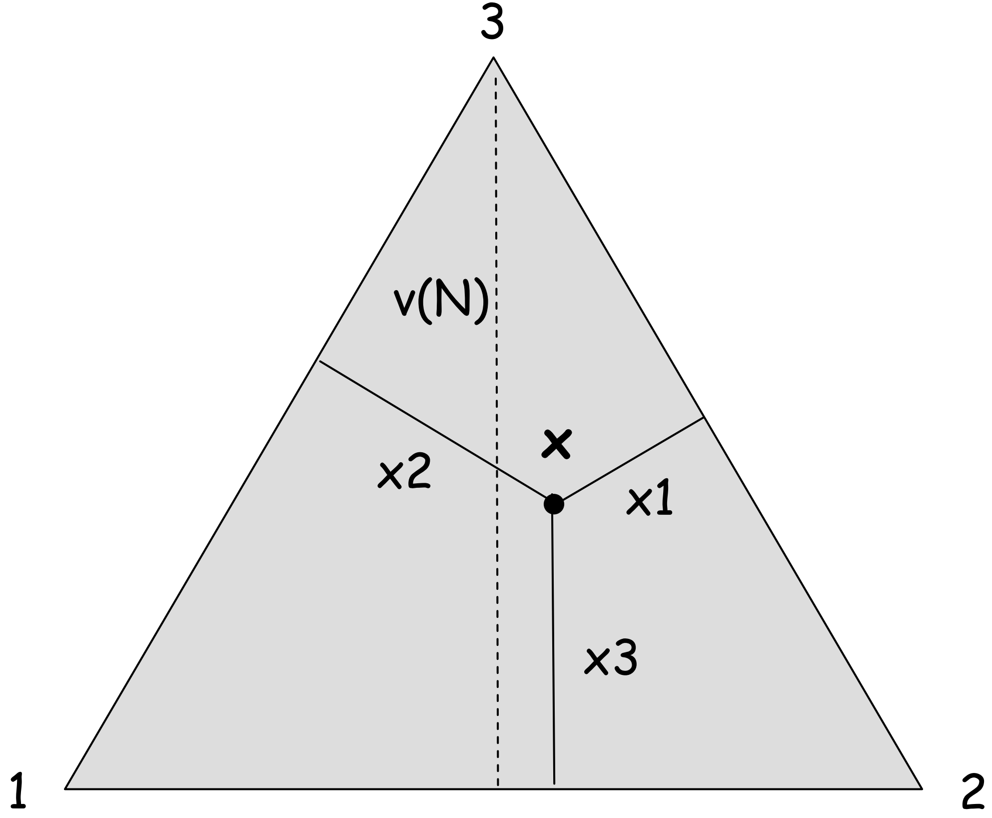
- Given a game in characteristic function form (N,v)
- an imputation is a payoff division...
  - i.e. a n-tuple of numbers **x**=(x1,x2,...,xn)
- with two reasonable properties
  - xi >= v(i) *(individual rationality)*
  - $\sum$xi >= v(N) *(collective rationality)*
- for superadditive games
  - $\sum$xi = v(N)

Imputation:
a graphical representation



a+b+c = h

Imputation: a graphical representation



- in general a n-dimensional simplex

Dominance

- An imputation **x** dominates an imputation **y** if there is some coalition S, s.t.
  - xi > yi for all i in S
    - **x** is more convenient for players in S
  - $\sum_{i \text{ in } S}$ xi <= v(S)
    - the coalition S must be able to enforce **x**

Dominance in example 1

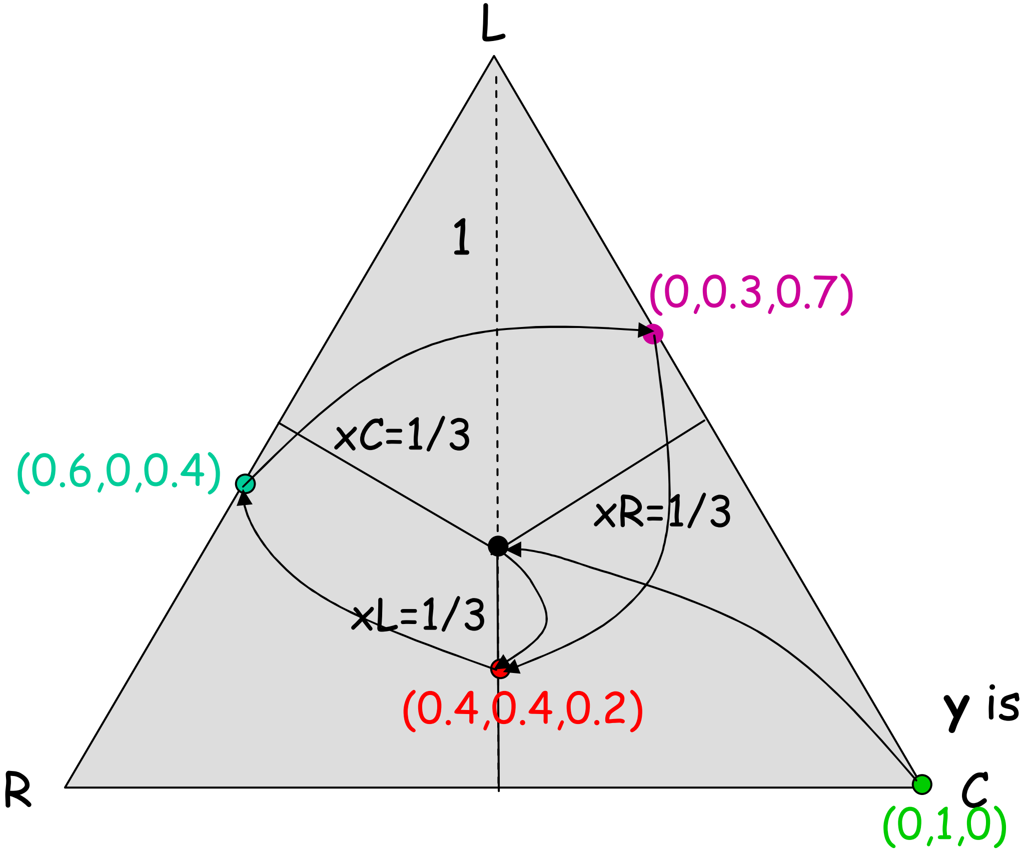
v(void)=0
v(R)=-4.40,
v(C)=-4.00,
v(L)=-1.43,
v(RC)=1.43,
v(RL)=4.00,
v(CL)=4.40,
v(RLC)=0

Normalization

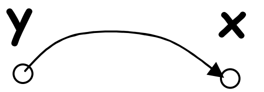
v(void) = 0
v(R) = 0,
v(C) = 0,
v(L) = 0,
v(RC) = 1,
v(RL) = 1,
v(CL) = 1,
v(RLC) =1

- Divide-the-dollar

Dominance in example 1



v(void)=0
v(R)=0, v(C)=0,
v(L)=0,
v(RC)=1,
v(RL)=1,
v(CL)=1,
v(RLC)=1

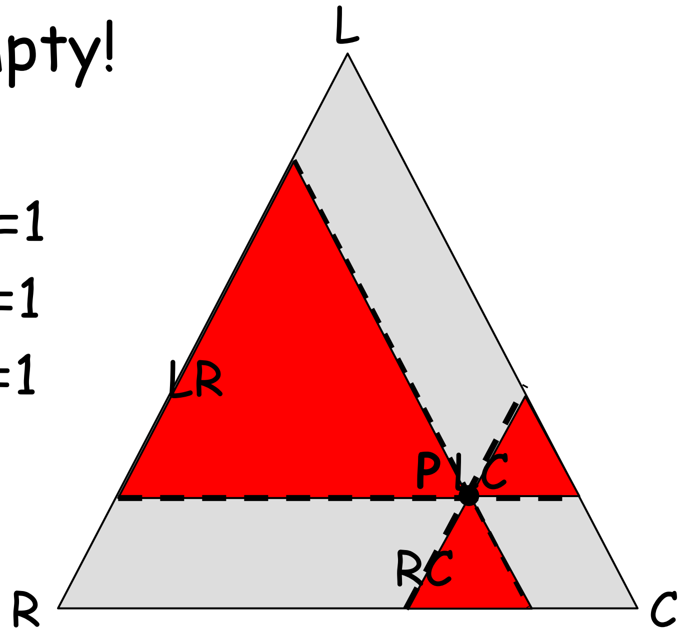


y is dominated by **x**

- Dominance is not transitive!

The Core

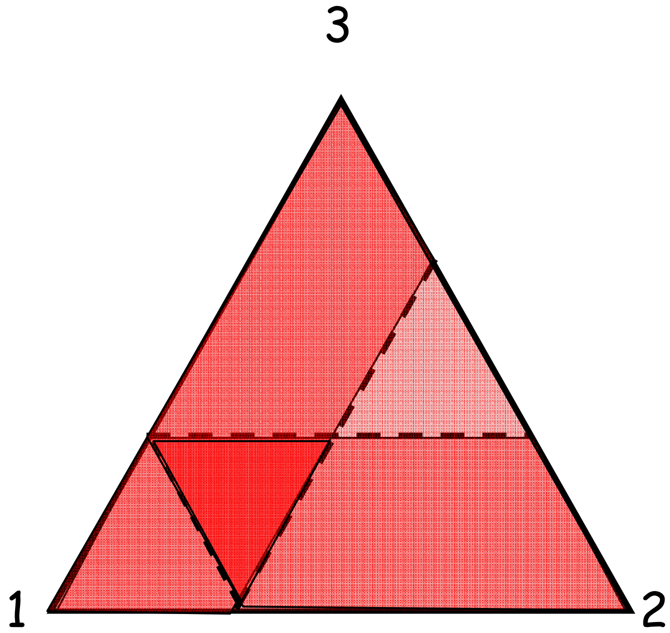
- The set of all undominated imputations,
  - i.e. the set of all imputations **x** s.t.
    for all S, $\Sigma_{i \text{ in } S}$ xi >= v(S)
- What about Divide-the-dollar?
  - the core is empty!
  - analitically
    - xR+xC>=v(RC)=1
    - xR+xL>=v(RL)=1
    - xL+xC>=v(LC)=1



P is dominated
by the imputations
in the red area

The Core

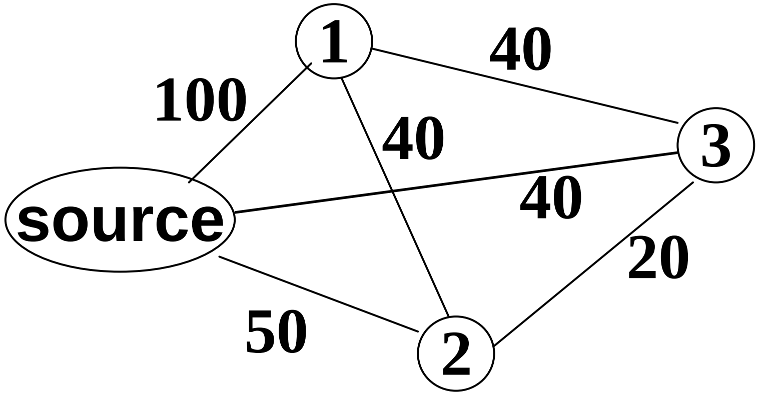
- What about Minimum Spanning Tree game?
  - v(void)= v(1) = v(2) = v(3)=0
  - v(12) = 60, v(13) = 60, v(23) = 30
  - v(123) = 90
- Analitically
  - x1+x2>=60, iff x3<=30
  - x1+x3>=60, iff x2<=30
  - x2+x3>=30, iff x1<=60

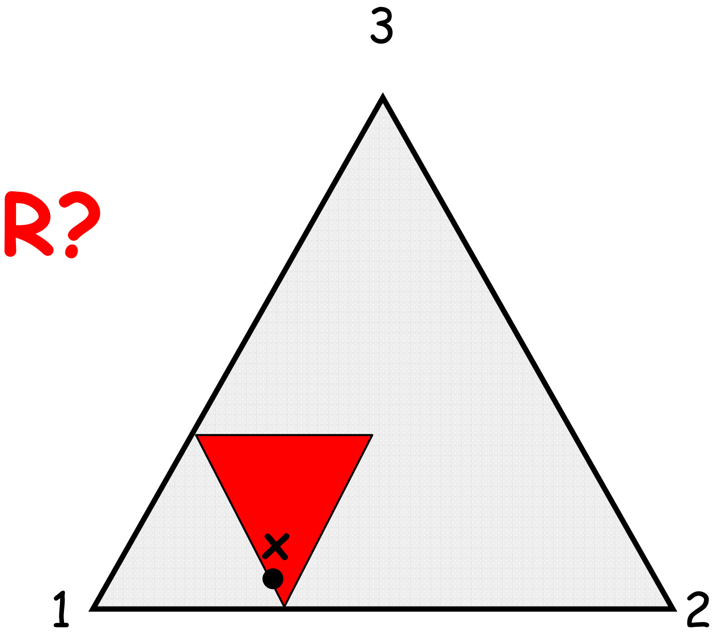


The Core

- Let's choose an imputation in the core: **x**=(60,25,5)
- The payoffs represent the savings, the costs under **x** are
  - c(1)=100-60=40,
  - c(2)=50-25=25
  - c(3)=40-5=35

FAIR?





The Shapley value

- Target: a fair imputation **k**
- Axioms
  - 1) if i and j have symmetric roles in v(), then ki=kj
  - 2) if v(S)=v(S-i) for all S, then ki=0
  - 3) if v and w are two games with the same player set and k(v) and k(w) the imputations we consider, then k(v+w)=k(v)+k(w)

    (Shapley value's weakness)
- Theorem: There is one and only one method of assigning such an imputation to a game

  (Shapley value's strength)

The Shapley value: computation

- Consider the players forming the grand coalition step by step
  - start from one player and add other players until N is formed
- As each player joins, award to that player the value he adds to the growing coalition
- The resulting awards give an imputation
- Average the imputations given by all the possible orders
- The average is the Shapley value **k**

The Shapley value: computation

- Minimum Spanning Tree game
  - v(void) = v(1) = v(2) = v(3)=0
  - v(12) = 60, v(13) = 60, v(23) = 30, v(123) = 90

		Value added by		
		1	2	3
Coalitions	123			
	132			
	213			
	231			
	312			
	321			
	avg			

The Shapley value: computation

- MST game
  - v(void) = v(1) = v(2) = v(3)=0
  - v(12) = 60, v(13) = 60, v(23) = 30, v(123) = 90

Value added by

Coalitions

	1	2	3
123	0	60	30
132			
213			
231			
312			
321			
avg			

The Shapley value: computation

- MST game
  - v(void) = v(1) = v(2) = v(3)=0
  - v(12) = 60, v(13) = 60, v(23) = 30, v(123) = 90

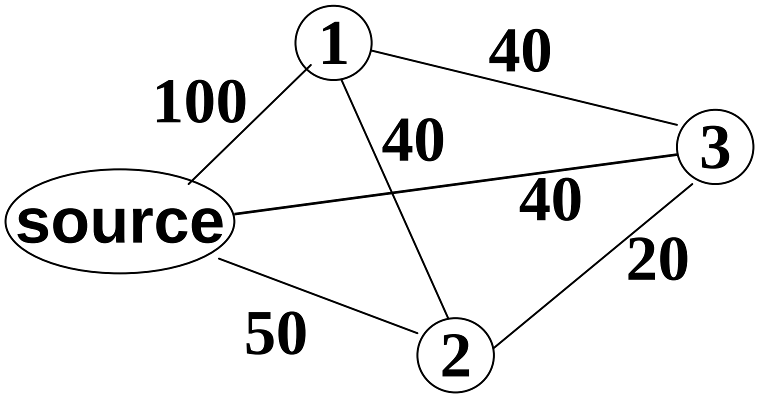
		Value added by		
		1	2	3
Coalitions	123	0	60	30
	132	0	60	30
	213	60	0	30
	231	60	0	30
	312	60	30	0
	321	60	30	0
	avg	**40**	**30**	**20**

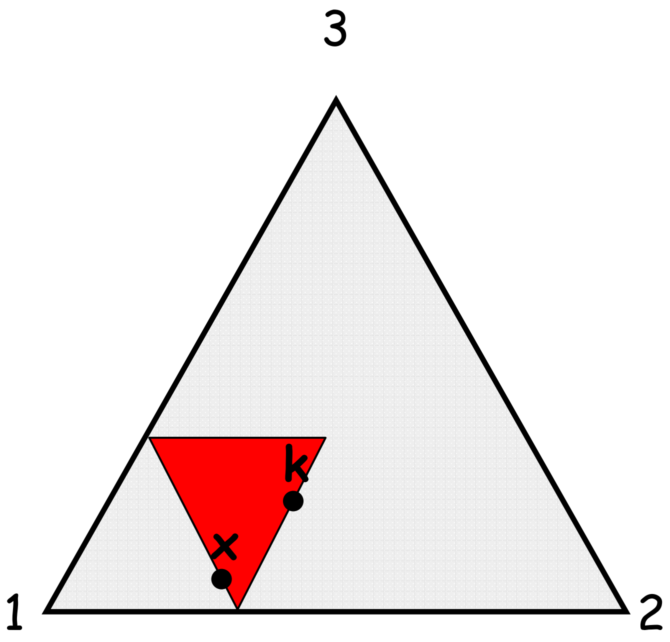
The Shapley value: computation

- A faster way
- The amount player i contributes to coalition S, of size s, is v(S)-v(S-i)
- This contribution occurs for those orderings in which i is preceded by the s-1 other players in S, and followed by the n-s players not in S
- ki = 1/n! $\sum_{S:i \text{ in } S}$ (s-1)! (n-s)! (v(S)-v(S-i))

The Shapley value: computation

- **k** = (40,30,20)
- the costs under **x** are
  - c(1)=100-40=60,
  - c(2)=50-30=20
  - c(3)=40-20=20





An application: voting power

- A *voting game* is a pair (N,W) where N is the set of players (voters) and W is the collection of winning coalitions, s.t.
  - the empty set is not in W (it is a losing coalition)
  - N is in W (the coalition of all voters is winning)
  - if S is in W and S is a subset of T then T is in W
- Also *weighted voting game* can be considered
- The Shapley value of a voting game is a measure of voting power (Shapley-Shubik power index)
  - The winning coalitions have payoff 1
  - The loser ones have payoff 0

An application: voting power

- The United Nations Security Council in 1954
  - 5 permanent members (P)
  - 6 non-permanent members (N)
  - the winning coalitions had to have at least 7 members,
  - but the permanent members had veto power
- A winning coalition had to have at least seven members including all the permanent members
- The seventh member joining the coalition is the pivotal one: he makes the coalition winning

An application: voting power

- 462 (=11!/(5!*6!)) possible orderings
- Power of non permanent members
  - (PPPPPN)N(NNNN)
  - 6 possible arrangements for (PPPPPN)
  - 1 possible arrangements for (NNNN)
  - The total number of arrangements in which an N is pivotal is 6
  - The power of non permanent members is 6/462
- The power of permanent members is 456/462, the ratio of power of a P member to a N member is 91:1
- In 1965
  - 5 permanent members (P)
  - 10 non-permanent members (N)
  - the winning coalitions has to have at least 9 members,
  - the permanent members keep the veto power
- Similar calculations lead to a ratio of power of a P member to a N member equal to 105:1

Other approaches

- Stable sets
  - sets of imputations J
    - internally stable (non imputations in J is dominated by any other imputation in J)
    - externally stable (every imputations not in J is dominated by an imputation in J)
  - incorporate social norms
- Bargaining sets
  - the coalition is not necessarily the grand coalition (no collective rationality)
- Nucleolus
  - minimize the unhappiness of the most unhappy coalition
  - it is located at the center of the core (if there is a core)
- Gately point
  - similar to the nucleolus, but with a different measure of unhappyness

Applications of cooperative game theory in Computer networks

- "The Shapley Value: Its Use and Implications on Internet Economics", Richard T.B. Ma, Dahming Chiu, John C.S. Lui, Vishal Misra and Dan Rubenstein, Allerton Conference on Communication, Control and Computing, September, 2008.